Introduction to Nuclear Physics

Physics 124

Solution Set 6

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Problem 22

In order to thermalize a neutron it must undergo multiple elastic collisions. Upon each interaction it will lose some energy which is transfered to the scattering particle. We will first work out in general a formula describing energy loss from multiple elastic scattering. Then examine the specific cases of scattering from ^{2}H, ^{12}C, and ^{238}U.

We begin the derivation using the basic principles, conservation of momentum and energy,

$$m_1\vec{v}_1 = m_1\vec{v}_1' + m_2\vec{v}_2' \tag{1}$$

$$m_1\vec{v}_1^2 = m_1\vec{v}_1'^2 + m_2\vec{v}_2'^2 \tag{2}$$

Where we define: m_1 is the particle being moderated

m_2 is the particle being scattered

$\vec{v}$ is the velocity of the particle before interacting

$\vec{v}'$ is the velocity of the particle after interacting

For simplicity, we assume head-on collisions. We can divide the energy equation by the momentum equation to obtain

$$\frac{m_2\vec{v}_2'^2}{m_2\vec{v}_2'} = \frac{m_1(\vec{v}_1^2 - \vec{v}_1'^2)}{m_1(\vec{v}_1 - \vec{v}_1')} \tag{3}$$

$$\vec{v}_2' = \frac{(\vec{v}_1 - \vec{v}_1')(\vec{v}_1 + \vec{v}_1')}{(\vec{v}_1 - \vec{v}_1')} = \vec{v}_1 + \vec{v}_1' \tag{4}$$

Now we can use this result to determine the energy lost by the particle being moderated, m_1 for each interaction. Using the conservation of momentum equation

$$m_1\vec{v}_1 = m_1\vec{v}_1' + m_2\vec{v}_1 + m_2\vec{v}_1', \tag{5}$$

$$\vec{v}_1' = \vec{v}_1\frac{(m_1 - m_2)}{(m_1 + m_2)}. \tag{6}$$

The maximum energy loss, for a head on collision, per interaction is simply

$$\delta E = \frac{m_1}{2}(v_1^2 - v_1'^2) = \frac{m_1}{2}v_1^2(1 - \left(\frac{(m_1 - m_2)}{(m_1 + m_2)}\right)^2), \tag{7}$$

$$\delta E = E_{initial}(1 - \left(\frac{(m_1 - m_2)}{(m_1 + m_2)}\right)^2). \tag{8}$$

We can rearrange this to find the final energy after n interactions

$$n = \frac{log\left(\frac{E_{final}}{E_{initial}}\right)}{log\left(\left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2\right)} \tag{9}$$

We then find that it takes the following number of collisions to moderate the neutron down to thermal energy on the order of $\frac{1eV}{40}$:

scattering from ^{2}H, n = 8

scattering from ^{12}C, n = 54

scattering from ^{238}U, n = 1074

Note: There is a more detailed way of making this calculation that is performed in Krane, section 12.2. In Krane's derivation he takes into account the average energy loss per interaction whereas I treated every interaction as a maximum energy loss.

Problem 23

The main thing to realize is that each fission event releases an average energy of 200 MeV for a ^{235}U produced chain reaction.

a) The average power of a research reactor is P = 10 MW. The number of fission events per second is then

$$n = \frac{P}{E} = \frac{10MW}{200MeV} = \frac{10^6 W}{200MeV(\frac{1.602\times10^{-13}J}{1MeV})} \approx 10^{17}\frac{1}{sec}. \tag{10}$$

b) The fuel burning rate can then be determined as follows, using the mass of ^{235}U,

$$FuelBurningRate = 10^{17}\frac{1}{sec}3.95 \times 10^{-22}g \approx 10^{-4}\frac{g}{sec}. \qquad (11)$$

Problem 24

The fuel used in nuclear reactors usually consists of a few percent of ^{235}U mixed with ^{238}U. ^{235}U has a half life of 7.038×10^8 years and ^{238}U has a half life of 4.47×10^9 years. Since ^{238}U is the most abundant element the initial activity of the fuel is primarily due to the ^{238}U.

When the fuel is being used in the reactor many radioactive isotopes are formed from the fission fragments at a relatively high rate, calculated above in question 23. These radioactive isotopes have various half-lives ranging from seconds to millions of years. The final state of the burnt fuel is that it is far more radioactive than the unburnt fuel. However the isotopes are now primarily beta and gamma emitters. By examining figure 13.30 in Krane we can get a rough idea of an effective half life for the burnt fuel. The initial half life appears to be a couple of years. So lets calculate a ratio of activities between the products and the fuel per cubic centimeter for 1 year of running,

$$\frac{A(products)}{A(fuel)} = \frac{\lambda_{products}N_{products}}{\lambda_{fuel}N_{fuel}}. \qquad (12)$$

In 1 year of running there are approximately

$$N_{products} = 10^{-4}\frac{grams}{sec} \times 3.154 \times 10^7 sec\frac{6.022 \times 10^{23}}{238 grams} \approx 8 \times 10^{24}, \qquad (13)$$

fission products formed by burning a total of

$$Mass = 10^{-4}\frac{grams}{sec} \times 3.154 \times 10^7 sec = 3154 grams, \qquad (14)$$

of uranium. The density of uranium is 19 grams per cubic centimeter so we have burnt 166 cubic centimeters of fuel. Now we can determine the number of products formed per cubic centimeter as $4.819 \times 10^{22} \frac{atoms}{cm^3}$. The number of atoms of fuel per cubic centimeter can be determined from the density of uranium and the molar mass,

$$N_{fuel} = \frac{6.022 \times 10^{23} atoms}{238 grams} \times 19 \frac{grams}{cm^3} = 4.807 \times 10^{22} \frac{atoms}{cm^3}. \tag{15}$$

Now we can determine the ratio of the activities per cubic centimeter,

$$\frac{A(products)}{A(fuel)} = \frac{\frac{ln2 \times 1year}{2years} 4.8 \times 10^{22} atoms}{\frac{ln2 \times 1year}{4.47 \times 10^9 years} 4.8 \times 10^{22} atoms} = \frac{4.47 \times 10^9}{2} \approx 2 \times 10^9. \tag{16}$$

So we can conclude that the products of the nuclear reactor are approximately 10^9 times more radioactive than the initial fuel. Also note that in the end the ratio of activities is dominated by the ratio of the half lives of the fuel and products and is independent of the power of the reactor.

More information on this topic can be found on the world wide web at www.incs.anl.gov as well as in Nuclear Energy by R.L. Murray, 1980.

Problem 25

We have an infinite spherically symmetric potential well with radius, a = 5 fm. Because of the spherical symmetry, our main concern is solving the radial Schroedinger equation which is

$$-\frac{\hbar^2}{2m}\Big(\frac{d^2R(r)}{dr^2} + \frac{2}{r}\frac{dR(r)}{dr}\Big) + \Big(\frac{l(l+1)\hbar^2}{2mr^2}\Big)R(r) = ER(r). \tag{17}$$

In the region where the potential is zero, the solutions to the radial equation are the spherical Bessel functions from M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions (New York: Dover, 1965):

$$j_0(kr) = \frac{sin(kr)}{kr}, \tag{18}$$

$$j_1(kr) = \frac{sin(kr)}{(kr)^2} - \frac{cos(kr)}{kr}, \tag{19}$$

$$j_2(kr) = \frac{3sin(kr)}{(kr)^3} - \frac{3cos(kr)}{(kr)^2} - \frac{sin(kr)}{kr}, \tag{20}$$

$$j_3(kr) = \left(\frac{15}{(kr)^4} - \frac{6}{(kr)^2}\right)sin(kr) + \left(\frac{15}{(kr)^3} + \frac{1}{kr}\right)cos(kr). \tag{21}$$

We must use these function combined with the boundary condition and the energy relation,

$$E = \frac{\hbar^2 k^2}{2m}, \tag{22}$$

to determine the energy of the first three radial excitations for l = 0,1,2,and 3.

At the boundary, a = 5 fm, the wave function must go to zero,

$$j_l(ka) = 0. \tag{23}$$

We must look up the first three zero crossing values in order to determine the value of k so we can calculate the energy of each excited state. From the Handbook of Mathematical Functions edited by M. Abramowitz and I.A. Stegun, 1965, we have,

for l = 0 the first three zeros occur at: ka = 3.14, 6.28, 9.42,

for l = 1 the first three zeros occur at: ka = 4.49, 7.73, 10.90,

for l = 2 the first three zeros occur at: ka = 5.76, 9.09, 12.32,

for l = 3 the first three zeros occur at: ka = 6.99, 10.41, 13.70.

From these values we can now calculate k and thus the energy of each state. The energy levels are,

E_0 = 8.20, 32.79, 73.77 MeV,

E_1 = 16.76, 49.68, 98.78 MeV,

E_2 = 27.58, 68.69, 126.19 MeV,

$E_3 = 40.62, 90.09, 156.04$ MeV.
Now what are the degeneracies of these levels? The complete wave function for the states also includes the spherical harmonics. Therefore the degeneracy of each level is identical to that of angular momentum in quantum mechanics. The degeneracies are then:
for l = 0 degeneracy = 1,
for l = 1 degeneracy = 3,
for l = 2 degeneracy = 5,
for l = 3 degeneracy = 7.
Of course we are ignoring the spin of the nucleon in this discussion of degeneracy.

Problem 26

The shell model is useful at predicting the ground state spin state of nuclei that have a single unpaired nucleon which lies above a closed shell. Alternatively, if a shell is one nucleon short of being full, we can consider it as a "hole." By filling in the levels on the shell model diagram on page 123 figure 5.6 of Krane, in some cases, we can determine the spin of the ground state.

Examining the list, we see that there are actually only four cases where we can make a prediction in this case:

$^{17}F : Z = 9, N = 8, I_{Shell} = 5/2, I_{Experiment} = 5/2$

$^{39}K : Z = 19, N = 20, I_{Shell} = 3/2, I_{Experiment} = 3/2$ (this is a case with a hole)

$^{87}Rb : Z = 37, N = 50, I_{Shell} = 5/2, I_{Experiment} = 3/2$ (what happens here is that the $1f$ shell fills first, leaving a hole in the $2p$ shell. Why this is so is beyond the material that we covered).

$^{171}Yb : Z = 70, N = 101, I_{Shell} = 5/2, I_{Experiment} = 1/2$ (this nucleus has A beyond the range of $A < 150$, where the model is expected to work well; see Krane, p. 125).

In conclusion we have been able to correctly predict a couple of the ground state spins, but our simple-minded approach should be applied with caution.

THANK YOU FOR THE GREAT SEMESTER !