Introduction to Nuclear Physics

Physics 124

Solution Set 2

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Problem 5

Let us start with order of magnitude estimates. We expect the quadrupole moment to have the form eQ $\sim$ er^2. Well $r \approx R_o A^{\frac{1}{3}}$ if we consider the valence charges are the source of the quadrupole moment. We can compute r and find that for light nuclei $eQ \sim 6 \times 10^{-30} em^2$ and for heavy nuclei $eQ \sim 50 \times 10^{-30} em^2$. Using the units of barns we can say that the quadrupole moment should fall somewhere in a range of 0.06 to 0.5 eb if it is present at all.

Now we will calculate the quadrupole moment, but first we should have an idea of the boundary conditions for our equation.

Assume that b is along the z-axis, and that the ellipsoid is obtained by revolution around the z-axis. Consider the following two limiting cases:

1) If we set b = 0 we have a pancake-like distribution. This would give us a quadrupole moment eQ $\sim$ -qa^2, where q is the total charge.

2) If we set a = 0 the charges are concentrated along the z-axis. We would then get a quadrupole moment eQ $\sim$ qb^2.

3) For a = b, we have a sphere and eQ = 0.

Our solution must match these limiting cases.

For calculating the quadrupole moment we use the following equation

$$eQ = \int (3z^2 - r_o^2)\rho(x')d^3x'. \qquad (1)$$

Working in cylindrical coordinates and pulling out the constant charge density we have

$$eQ = \rho \int (3z^2 - r_o^2) r dr d\phi dz. \qquad (2)$$

The integration limits for the ellipsoid can be written as

$$\phi : 0 \rightarrow 2\pi \qquad z : -b \rightarrow b \qquad r : 0 \rightarrow r'. \qquad (3)$$

We cn find r' from the equation of an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1. \tag{4}$$

Setting $r^2 = x^2 + y^2$ we find r' to be

$$r' = a(1 - \frac{z^2}{b^2})^{\frac{1}{2}}. \tag{5}$$

We must be careful to substitute in r in cylindrical coordinates for r_o which is the distance from the origin in equation (1) and equation (2),

$$r_o^2 = z^2 + r^2. \tag{6}$$

Then we have to solve the following integral

$$eQ = \int_{z=-b}^{b} \int_{r=0}^{r'} \int_{\phi=0}^{2\pi} (2z^2 - r^2) r d\phi dr dz. \tag{7}$$

Integrating over r and ϕ we obtain

$$eQ = 2\pi\rho \int_{-b}^{b} \left(a^2 z^2 \left(1 - \frac{z^2}{b^2}\right) - \frac{a^4}{4}\left(1 - \frac{z^2}{b^2}\right)^2 \right) dz. \tag{8}$$

Using Mathematica, Maple, or anything else you like using to solve this we find the quadrupole moment to be (can also be done by hand without much work)

$$eQ = \frac{8\pi\rho a^2 b}{15}(b^2 - a^2), \tag{9}$$

or replacing the charge density by the total charge q we have

$$\rho = \frac{q}{\frac{4}{3}\pi a^2 b}, \tag{10}$$

$$eQ = q\frac{2}{5}(b^2 - a^2). \tag{11}$$

Which satisfies our limiting cases stated above.

Problem 6

The formation of daughter nuclei and their subsequent decay is the topic of this problem. The parent nuclei are of type A. These decay to produce nuclei B, the daughter nuclei, which are also unstable. The daughter can then decay to stable nuclei C. We are examining one of the simpler examples of a nuclear decay chain. A more complicated example would be the decay of Thorium-228 in which there are about eigth decay steps before a stable nucleus is encountered. The decay chain also includes alpha, beta, and gamma decay. The understanding of nuclear decay chains was recently used by my colleagues at the 88" cyclotron to identify the new elements 118 and 116, as yet unnamed.

a) First we will derive the equation governing the activity of the daughter nuclei. We begin with some basic definition of exponential decay. The parent nucleus decays by the following relation

$$N_A(t) = N_o e^{-\lambda_A t}. \tag{12}$$

Now, for the number of the daughter nuclei we can write

$$dN_B = \lambda_A N_A dt - \lambda_B N_B dt. \tag{13}$$

By guessing at a solution to the above differential equation and using the fact that at t = 0 no daughter nuclei exist we can solve the equation for the number of daughter nuclei present as a function of time. Another way to solve the equation would be to use the general rule for the solution of inhomogeneous differential equations. We find that a good guess for a solution is

$$N_A = C_1 e^{-\lambda_A t} + C_2 e^{-\lambda_B t}. \tag{14}$$

Along with the initial condition $N_B(0) = 0$ we find C_1 = -C_2. Now we have

$$N_B = C_1 e^{-\lambda_A t} - C_2 e^{-\lambda_B t}. \tag{15}$$

Using the above equation for N_B we find

$$-\lambda_A C_1 + \lambda_B C_1 = \lambda_A N_o - \lambda_B (C_1 - C_1). \tag{16}$$

Solving for C_1 we can then find N_B

$$C_1 = N_o \frac{\lambda_A}{\lambda_B - \lambda_A} \tag{17}$$

$$N_B(t) = N_o \Big(\frac{\lambda_A}{\lambda_B - \lambda_A} \Big) (e^{-\lambda_A t} - e^{-\lambda_B t}). \tag{18}$$

The activity of the daughter is then given by the following equation

$$A_B(t) \equiv \lambda_B N_B(t) = N_o \Big(\frac{\lambda_B \lambda_A}{\lambda_B - \lambda_A} \Big) (e^{-\lambda_A t} - e^{-\lambda_B t}). \tag{19}$$

We now need to substitute in the lifetimes so we can see how the activity depends on the half-life of the parent and daughter nuclei.

$$\lambda_1 = \frac{ln(2)}{\tau_A} \qquad \lambda_2 = \frac{ln(2)}{\tau_B} \tag{20}$$

The activity of the daughter is now represented by

$$A_B(t) \equiv \lambda_B N_B(t) = N_o \Big(\frac{ln(2)}{\tau_A - \tau_B} \Big) (e^{-\frac{ln(2)}{\tau_A} t} - e^{-\frac{ln(2)}{\tau_B} t}). \tag{21}$$

b) Now we will examine the various cases mentioned at the beginning.

case 1 $\tau_A \gg \tau_B$

Here we see that the activity is given by

$$A_B(t) = N_o \Big(\frac{ln(2)}{\tau_A} \Big) (1 - e^{-\frac{ln(2)}{\tau_B} t}). \tag{22}$$

This behaviour is shown graphically on page 170 in Krane Fig. 6.5. In this case the parent acts as a source that eventually stops leaving the daughter nuclei to exponentially decay. The turning point at the top is called the secular equilibrium value. This is when the decay rate equals the production rate.

case 2 $\tau_A \ll \tau_B$

We now find that the activity of the daughter is

$$A_B(t) = N_o\left(\frac{ln(2)}{\tau_B}\right)e^{-\frac{ln(2)}{\tau_B}t}. \tag{23}$$

So we will see a rapid rise in the activity of the daughter initially which corresponds to the short lifetime of the parent. Then the activity is described by the above equation dominated by the lifetime of the daughter.

case 3 $\tau_A \approx \tau_B$

If the two half-lifes are almost equal we have to make a power series expansion of the exponential terms. We end up only keeping the first non-zero value and we set $\tau = \tau_A = \tau_B$ to get

$$A_2(t) \approx N_o\left(\frac{ln(2)}{\tau}\right)^2 t. \tag{24}$$

This result shows that the acivity of the daughter increases linearly in time. However it is only a rough approximation as we omitted the higher order terms. If we included all the terms we would see that the long term behaviour will reduce the activity of the daughter down to zero.

Problem 7

Carbon dating is a technique used to estimate the age of a once living object. The technique makes use of the knowledge that there is a relatively constant ratio of ^{14}C to ^{12}C in the Earth's atmosphere. The carbon is found in the atmosphere in the form of CO_2. The carbon is removed from the atmosphere by plants and is used for growth. The percentage of ^{13}C is 1.11% the rest being ^{12}C, 98.89%. Once a plant dies it no longer fixes carbon from the atmosphere and the amount present gradually decays away. The half-life of ^{14}C is 5730 years. By comparing the specific activity of two different age materials we can determine the age of one piece relative to the other. Here we have a sample that has a specific activity of 5.3 dpm that is 0 years old and another which has a specific activity of 2.1 dpm. So how old is the wood? We set the specific activities equal to each other

$$A_1 e^{-\frac{ln2}{\tau_{\frac{1}{2}}} t_1} = A_2 e^{-\frac{ln2}{\tau_{\frac{1}{2}}} t_2}. \tag{25}$$

Using $t_1 = 0$, $A_1 = 2.1$, and $A_2 = 5.3$ we find

$$2.1 = 5.3 e^{-\frac{ln2}{\tau_{\frac{1}{2}}} t_2}. \tag{26}$$

Solving for t_2 we find that the wood is 7653 years old.

Problem 8

a) Alpha decay can only occur if the Q value

$$Q = (m_X - m_{X'} - m_\alpha) \tag{27}$$

for a given nucleus, X, is positive. When Q is positive energy is released in the form of kinetic energy of the alpha particle, α, and the recoiling nucleus, X'. By examining the Q value for alpha decay around N=56 one finds that the Q value is negative. Thus it is uncommon for for these nuclei to undergo alpha decay.

Another way to see this is to examine the graph on page 67 in Krane. There the binding energy per nucleon is at a maximum for nuclei around N=56. It would not be energetically favorable to form an alpha particle and leave the newly formed nucleus in a less bound final state.

b) A similar argument holds here. The Coulomb barrier is lower for proton emission than for alpha emission, roughly half the size. However the Q value is still negative for proton emission in general thus forbidden by energy conservation.