PHY 137A (D. Budker) Solutions to Midterm Review Problems
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1. We will consider a hydrogenic ion with a Carbon-12 core, so $Z = 6$ and $A = 12$.

   (a) Describe the kind of radiation I would need to use to completely ionize this ion in its ground state. Calculate the maximum wavelength of radiation (in nm) which could be used for this purpose and its energy per photon (in eV).

   Solution: The first Bohr radius in hydrogenic carbon ion can be found by just noting that the only modification to the hydrogen problem from having a carbon core is that we need to modify all terms involving $e^2 \longrightarrow Ze^2$. In particular, this means that the ionization energy must be modified as

$$-E_1 = \frac{me^4}{2\hbar^2} \longrightarrow Z^2 E_1 = \frac{mZ^2 e^4}{2\hbar^2} = 36 \cdot 13.6 \text{ eV} = 490 \text{ eV}. \tag{1.1}$$

   Thus, we would need to use radiation with wavelength

$$\lambda = \frac{hc}{E} = \frac{1.24 \times 10^3 \text{ eV} \cdot \text{ nm}}{490 \text{ eV}} = 2.53 \text{ nm}. \tag{1.2}$$

   These would correspond to soft X-Rays.

   (b) Find the energy (in eV), frequency (in Hz), and wavelength (in nm) of the analogue of the Paschen-α line for this hydrogenic ion.

   Solution: The Paschen-α line is the transition from $n_i = 4$ to $n_f = 3$. Using the results from the previous section, we can easily find the energy by noting that it is:

$$E = -E_1 \left(\frac{1}{3^2} - \frac{1}{4^2} \right) = 490 \text{ eV} \cdot \left(\frac{7}{144} \right) = 23.8 \text{ eV}. \tag{1.3}$$

   The wavelength is then just dilated by a factor of $\frac{144}{7}$, which means that

$$\lambda = 2.53 \text{ nm} \cdot \frac{144}{7} = 52.0 \text{ nm}. \tag{1.4}$$

   Finally, the frequency can be calculated remembering that $\nu = E/h$, and that $1/h = 2.42 \times 10^{14}$ Hz $\cdot$ eV^{-1} so we have

$$\nu = 23.8 \text{ eV} \cdot 2.42 \times 10^{14} \text{ Hz} \cdot \text{ eV}^{-1} = 5.76 \times 10^{15} \text{ Hz} \tag{1.5}$$

   (c) Calculate the approximate Bohr radius of the electron before and after this transition.

   Solution: The Bohr radius of the ground state in a hydrogenic ion must be modified as

$$a = \frac{\hbar^2}{me^2} \longrightarrow a/Z = \frac{\hbar^2}{Zme^2} = a_0/6 = 8.82 \times 10^{-10} \text{ cm}, \tag{1.6}$$

   Now, for an excited state, we can find the answer at least approximately by considering the fact that these orbitals have radial wavefunctions of the form:

$$R_{nl}(r) = r\rho^{l-1} e^{-\rho} v(\rho) \tag{1.7}$$

   where $\rho = \frac{rZ}{na_0}$. From the form of this wavefunction, we can immediately see that the exponential term ensures that the bulk of the wavefunction is supported in the region with $\rho < 1$, or $r < na$. Thus, we expect that the Bohr radius of the n-th orbital, which is the expectation value of r in that state, will be an integral over a function whose support is going to be to largely contained within the region $r \leq na_0/Z$. This suggests that the Bohr radius should in fact be approximately equal to na_0/Z, so we find that the Bohr radius before and after are just:

$$a_i = 4a_0/6 = 3.53 \times 10^{-9} \text{ cm} \tag{1.8}$$

$$a_f = 3a_0/6 = 2.65 \times 10^{-9} \text{ cm}. \tag{1.9}$$

(d) What be the change in frequency (in MHz) of this line due to the isotope effect if we had a Carbon-14 ion (with $A = 14$ and the same Z) instead of a Carbon-12 ion?

Solution: The change in frequency can be found simply by finding the difference in effective mass in the two situations, which is the only difference between the two cases. This is just:

$$\mu_{14} - \mu_{12} \approx \frac{m_e 14 m_p}{m_e + 14 m_p} - \frac{m_e 12 m_p}{m_e + 12 m_p} \approx m_e \frac{2 \cdot m_e}{12 \cdot 14 m_p} = m_e \frac{1}{7} \frac{9.11 \times 10^{-31}\text{ g}}{1.67 \times 10^{-27}\text{ g}} = 7.79 \times 10^{-5} \cdot m_e \tag{1.10}$$

Thus, we get a difference in frequency which is just

$$\begin{aligned} \Delta\nu &= \frac{\Delta E}{h} = \frac{1}{\hbar}\left(\frac{\mu_{14} Z^2 e^4}{2\hbar^2} - \frac{\mu_{12} Z^2 e^4}{2\hbar^2}\right)\left(\frac{1}{3^2} - \frac{1}{4^2}\right) \\ &= \frac{\mu_{14} - \mu_{12}}{m_e h} \cdot \left(\frac{m_e Z^2 e^4}{2\hbar^2}\right)\left(\frac{7}{144}\right) \\ &= 7.79 \times 10^{-5} \cdot \frac{23.8\text{ eV}}{h} = 7.79 \times 10^{-5} \cdot 5.76 \times 10^{15}\text{ Hz} = 4.49 \times 10^{5}\text{ MHz} \end{aligned} \tag{1.11}$$

(e) Suppose that I now apply various external fields and consider all relativistic (and other - excluding spin!) corrections such that I totally break *all* the degeneracies in the spectrum of the Carbon-12 Hydrogenic ion, but only slightly. That is, suppose that the correction to the energy of each state in the spectrum is different, but small. Then, if I use a very accurate spectrometer, I would find that the single Paschen-α line has actually split into many different lines. How many lines would I see?

Solution: Thus, each of the energies E_{nlm} are now different, though all the energies with the same n are still close to eachother. In this case, each line I would see would be coming from a transition from one the $n_i^2 = 4^2 = 16$ states with $n_i = 4$ to one of the $n_f^2 = 3^2 = 9$ states with $n_f = 3$. This would just give me $9 \cdot 16 = 144$ lines.

(f) Find all transitions that are visible.

Solution: In terms of energies of transitions, the visible range correspond to transitions with energies between 1.8 eV and 3.1 eV. To do this problem, we will start by finding two limiting cases. First, we need to find the first α line with a small enough energy to be in the visible range, and then second, the largest n_f such that for n_i very large, the energy of the transition is large enough to be in the visible spectrum. The smallest alpha line is found finding the smallest n_f such that:

$$3.1\text{ eV} \geq 490\text{ eV}\left(\frac{1}{n_f^2} - \frac{1}{(n_f+1)^2}\right) \Longrightarrow n_f \geq 7 \tag{1.12}$$

(which I did by trial and error). The largest n_f such that the transition has a large enough energy to be visible is found by considering the largest value of n_f which satisfies (where we take $n_i \to \infty$):

$$1.8\text{ eV} \leq 490\text{ eV}\left(\frac{1}{n^2}\right) \Longrightarrow n_f \leq 16 \tag{1.13}$$

Thus, we have found all of the allowed values of n_f which might lead to light in the visible spectrum. We now simply need to compute, for each value of n_f, the range of allowed values of n_i which would lead to a transition in the visible spectrum. This can be done with the help of a calculator or spreadsheet. Essentially, given a value of n_f, one is looking for all values of n_i such that

$$\frac{1.8}{490} = 3.673 \times 10^{-3} \leq \left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \leq \frac{3.1}{490} = 6.327 \times 10^{-3} \tag{1.14}$$

The solution is:

$$\begin{aligned} n_f &= 7 & n_i &= 8 \\ n_f &= 8 & n_i &= 9, 10 \\ n_f &= 9 & n_i &= 11, 12 \\ n_f &= 10 & n_i &= 13, 14, 15, 16 \\ n_f &= 11 & n_i &= 15 - 22 \\ n_f &= 12 & n_i &= 18 - 40 \\ n_f &= 13 & n_i &= 22 - \infty \\ n_f &= 14 & n_i &= 27 - \infty \\ n_f &= 15 & n_i &= 36 - \infty \\ n_f &= 16 & n_i &= 65 - \infty \end{aligned}$$

These are all the transitions in the visible range!

2. (This is the last problem on my midterm review notes, which I left as an exercise). Consider the following state (we assume that $m + 1 \leq l$):

$$\Psi = \frac{1}{\sqrt{2}} \left(|lm\rangle + |lm+1\rangle \right). \tag{1.15}$$

Compute $\langle L_z \rangle$, $\langle L_x \rangle$, and $\langle L_x^2 \rangle$ in this state.

Solution: The solution for $\langle L_x \rangle$ is in my review notes, and I refer you to them for the solution. The answer is

$$\langle L_x \rangle = \frac{\hbar}{2} \sqrt{l(l+1) - m(m+1)} \tag{1.16}$$

The answer for $\langle L_y \rangle$ is nearly identical, except that the two terms which are identical and left at the end of the day have the opposite signs, and actually cancel, so $\langle L_y \rangle = 0$. I will repeat the steps used to derive this result for completeness. We use the properties of the angular momentum ladder operators and the states $|lm\rangle$ to do this computation. In particular, we use the following facts about the ladder operators,

$$L_\pm = L_x \pm i L_y \tag{1.17}$$

$$L_x = \frac{1}{2}(L_+ + L_-) \tag{1.18}$$

$$L_y = \frac{1}{2i}(L_+ - L_-) \tag{1.19}$$

$$L_\pm |lm\rangle = \hbar \sqrt{l(l+1) - m(m \pm 1)} |lm \pm 1\rangle \tag{1.20}$$

$$\tag{1.21}$$

and the following fundamental property of the states $|lm\rangle$

$$\langle l'm' | lm \rangle = \delta_{ll'} \delta_{mm'}. \tag{1.22}$$

Now, using the ladder operator relations in Eqns. (1.20) and the orthonormality relation (1.22), first consider:

$$\langle lm | L_\pm | lm \rangle = \hbar \sqrt{l(l+1) - m(m \pm 1)} \langle lm | lm \pm 1 \rangle = 0 \tag{1.23}$$

$$\langle lm | L_\pm^2 | lm \rangle \propto \langle lm | lm \pm 2 \rangle = 0. \tag{1.24}$$

Using these facts we clearly see that:

$$\langle lm|L_y|lm\rangle = \langle lm|\frac{1}{2i}(L_+ - L_-)|lm\rangle = 0 \tag{1.25}$$

This means that:

$$\begin{aligned}
\langle\Psi|L_y|\Psi\rangle &= \frac{1}{2}\left(\langle lm| + \langle lm+1|\right)(L_y)\left(|lm\rangle + |lm+1\rangle\right) \\
&= \frac{1}{2}\left(\langle lm|L_y|lm\rangle + \langle lm+1|L_y|lm\rangle + \langle lm|L_y|lm+1\rangle + \langle lm+1|L_y|lm+1\rangle\right) \\
&= \frac{1}{2}\left(\langle lm+1|L_y|lm\rangle + \langle lm|L_y|lm+1\rangle\right).
\end{aligned} \tag{1.26}$$

Now, writing $L_y = \frac{1}{2i}(L_+ - L_-)$ and using the relation Eqn. (1.20) for the action of the ladder operators on the state $|lm\rangle$, we find that:

$$\begin{aligned}
\langle\Psi|L_y|\Psi\rangle &= \frac{1}{2}\left(\langle lm+1|\frac{1}{2i}(L_+ - L_-)|lm\rangle + \langle lm|\frac{1}{2i}(L_+ - L_-)|lm+1\rangle\right) \\
&= \frac{1}{4i}\left(\langle lm+1|L_+|lm\rangle - \langle lm+1|L_-|lm\rangle + \langle lm|L_+|lm+1\rangle - \langle lm+1|L_-|lm+1\rangle\right) \\
&= \frac{\hbar}{4i}(\sqrt{l(l+1)-m(m+1)}\langle lm+1|lm+1\rangle - \sqrt{l(l+1)-m(m-1)}\langle lm+1|lm-1\rangle \\
&+ \sqrt{l(l+1)-(m+1)((m+1)+1)}\langle lm|lm+2\rangle - \sqrt{l(l+1)-(m+1)((m+1)-1)}\langle lm|lm\rangle) \\
&= \frac{\hbar}{4i}\left(\sqrt{l(l+1)-m(m+1)} - \sqrt{l(l+1)-(m+1)m}\right) = 0.
\end{aligned} \tag{1.27}$$

Calculating L_x^2 is actually easier than this calculation, since we can write this expectation value as

$$\langle L_x^2\rangle = \langle\frac{1}{4}\langle(L_+ + L_-)(L_+ + L_-)\rangle = \frac{1}{4}\langle(L_+^2 + L_-L_+ + L_+L_- + L_-^2)\rangle. \tag{1.28}$$

The terms involving L_+^2 and L_-^2 vanish, as they raise (lower) the value of m twice giving (for some constants A and B)

$$L_\pm^2|\Psi\rangle = A|lm\pm 2\rangle + B|l(m+1\pm 2)\rangle, \tag{1.29}$$

which is orthogonal to $|\Psi\rangle$. Now, using the fact that

$$L_\pm L_\mp = (L_x \pm iL_y)(L_x \mp iL_y) = L_x^2 + L_y^2 \mp i[L_x, L_y] = \mathbf{L^2} - L_z^2 \pm \hbar L_z, \tag{1.30}$$

and the fact that $\mathbf{L^2}$ and L_z^2 are diagonal in the $|lm\rangle$ basis, we have

$$\begin{aligned}
\langle L_x^2\rangle &= \frac{1}{4}\langle(L_-L_+ + L_+L_-)\rangle = \frac{1}{2}\langle\Psi|(\mathbf{L^2} - L_z^2)|\Psi\rangle \\
&= \frac{1}{2}\left(\langle lm| + \langle lm+1|\right)(\mathbf{L^2} - L_z^2)\left(|lm\rangle + |lm+1\rangle\right) \\
&= \frac{1}{2}\left(\langle lm|(\mathbf{L^2} - L_z^2)|lm\rangle + \langle lm+1|(\mathbf{L^2} - L_z^2)|lm+1\rangle\right) \\
&= \frac{\hbar^2}{2}\left(l(l+1) - m^2 + l(l+1) - (m+1)^2\right) = \frac{\hbar^2}{2}\left(2l(l+1) - (2m^2 - 2m - 1)\right).
\end{aligned} \tag{1.31}$$