PHY 137A (D. Budker) Final Review Problems
TA: Uday Varadarajan

Note: These problems are designed to be significantly more difficult than what you'll actually see on the final!

1. **Phonons in a Crystal:** (Note: Much of this problem is background info that you absolutely do NOT need for the test, so you can skim the bulk of the problem and get to the end to do the computations if you're in a rush!) We consider sound waves in a crystal. The elementary excitations of such sound waves are called phonons (in analogy with photons as the elementary excitations of electromagnetic waves).

   We consider a simple one dimensional lattice with two different atoms in every unit cell of the lattice. It can be shown that the dispersion relation for small oscillations of such a lattice (which is what sound waves really are) actually has two separate "branches" called the acoustic and optical branches. What is meant by this is that there are actually two different kinds of "sound waves" in such a crystal. The acoustic branch is characterized by the fact that the two atoms in a unit cell tend to move as a single unit. Thus, there can be low frequency, long wavelength oscillations where all the unit cells near a given unit cell are moving slowly in the same direction, but there is a gradual shift in the motion with a very long wavelength compared to a. It is these oscillations that we hear as sound waves. However, there is also the possibility that the two atoms in a unit cell are actually moving out of phase with each other, and the sound wave actually consists of oscillations within the unit cell itself! This kind of wave can have a very long wavelength (as all unit cells can be doing the same kind of oscillation for a large section of the crystal), but the frequency of the oscillation is actually very high, since the motion is restricted to remain within the unit cell. It turns out that these oscillations interact strongly with light and are largely responsible for the optical properties of crystals! We will consider the dispersion relations for these two kinds of phonons.

   We model the lattice by an infinite chain of balls and springs, such that there are two different kinds of balls (of the same mass M) connected to each other by springs in each unit cell of length a. We suppose that the spring that connects the two balls in each unit cell has spring constant K while the spring that connects each unit cell to the other unit cell has spring constant G. This is just an infinite set of coupled harmonic oscillators, and one can find wave-like solutions to this problem with wavelengths $\lambda = \frac{2\pi}{k} > a$. These wave-like solutions have a dispersion relation given by the square root of:

$$\omega^2 = (K+G)/M \pm (1/M) * \sqrt{K^2 + G^2 + 2KGcoska}. \tag{0.1}$$

   (a) Which branch is which? Hint: Consider the dispersion relation for small k.

   (b) For the acoustic branch in the limit of small k, what is the speed of sound? For small k, do sound waves disperse?

   (c) Take the limit that $K \gg G$, which corresponds to imagining that the crystal is actually up of diatomic molecules, each losely bound to eachother. Show that the optical branch has a constant dispersion relation in this limit. Thus, optical phonons of different wavelengths all have the same frequency. What does this frequency correspond to physically?

   (d) In the same limit as the previous part, calculate the dispersion relation for the acoustic branch, find the group velocity and phase velocity, and comment on whether such waves disperse or not.

2. We'll model a localized particle with some momentum p_0 as a Gaussian wave packet in momentum space with some momentum spread Δp,

$$\phi(p) = C \exp\left[-\frac{(p-p_0)^2}{2\Delta p^2}\right] \tag{0.2}$$

Using the fact that the time dependent position space wave function is

$$\Psi(x,t) \propto \int e^{i(px - E_p t)/\hbar} \phi(p) dp \tag{0.3}$$

where $E_p = \frac{p^2}{2m}$, and neglecting any normalizations and constants,

(a) Find Δx as a function of time. This can be found by considering the width of the packet when $|\psi(x,t)|^2$ falls off from its maximum to $1/e$ of its original value.

(b) At very small times, show that such a wavepacket starts off saturating the uncertainty relation (i.e. that $\Delta x \Delta p \approx \hbar$).

(c) Suppose that the particle is a free electron, initially localized to an atomic distance of an $\mathring{A}$. How long would it take for the electron to delocalize to a distance of $2\mathring{A}$?

3. We consider an electron in Hydrogen in a $2p_{3/2}$ state (we use spectroscopic notation nl_j, so this is a state with $n = 2$, total angular momentum $3/2$, and $l = 1$). Suppose that we've measured the projection of its total angular momentum in the z direction and got the value $\hbar/2$.

   (a) Find the energy of this state (including relativistic corrections).

   (b) Construct the matrices J_x, J_y, and J_z acting on this state. Hint: Use the abstract angular momentum relations and $J_\pm = J_x \pm iJ_y$.

   (c) Find $\langle J_x \rangle$ and $\langle J_y \rangle$.

4. (Problem 6.13 in Griffiths) Find the (lowest-order) relativistic correction to the energy levels of the one-dimensional harmonic oscillator. **Hint:** Use raising and lowering operators and the abstract formulation of the harmonic operator to do this problem. See Problem 2.37 in Griffiths.

5. Consider a particle in the ground state of the infinite square well. Suppose that at time $t = 0$ we suddenly perturb the system by placing a delta function bump of the form

$$H' = \alpha\delta(x - a/2) \tag{0.4}$$

   in the middle of the well. Assume that the bump is sufficiently small so that perturbation theory is valid.

   (a) What exactly does sufficiently small mean here, quantitatively?

   (b) What is the energy of the ground state of the perturbed system? What about the energy of the excited states?

   (c) If a measurement of energy is done immediately after the bump is introduced, what is the probability (to first order) that the particle will end up in the n-th excited state of the system?