PHY 137A (D. Budker) Midterm 2 Solutions
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1. (a) What is the wavelength (in nm) of the Lyman-alpha light in hydrogen?

   Solution: The Lyman-alpha transition is the transition between the $n_i = 2$ state and the $n_f = 1$ state in atomic hydrogen. The wavelength of the transition is given by:

   $$\frac{1}{\lambda} = R\left(\frac{1}{n_i^2} - \frac{1}{n_f^2}\right) = 1.097 \times 10^7 \text{ m}^{-1} \cdot \left(1 - \frac{1}{4}\right) = 8.23 \times 10^6 \text{ m}^{-1} \tag{2.1}$$

   Inverting this, we find that

   $$\lambda = 122 \text{ nm}. \tag{2.2}$$

   (b) Suppose you have to build a laser light source that can directly excite this transition. You can make use of two well-developed technologies: i) Dye lasers that can produce direct output at any desired wavelength in the visible range, and ii) Frequency doubling in non-linear optical crystals. How many successive frequency doubling stages will it take to produce the desired source?

   Solution: The visible spectrum consists of wavelengths from 400 nm to about 700 nm. Thus, I would use light at 4×122 nm = 488 nm produced by the dye laser and frequency double it twice. This would quadruple its frequency, which would reduce its wavelength by a factor of four, thereby giving me the desired light source with a wavelength of 122 nm.

2. (a) What is the first Bohr radius in Hydrogen (in cm)? Explain precisely what we mean by "radius" in this context.

   Solution: The radial wave function for the ground state of Hydrogen is:

   $$R_{10}(r) = const \cdot exp(-r/a), \tag{2.3}$$

   where a is the first Bohr radius. The expectation value for the radius of the electron in the ground state in Hydrogen can be easily calculated to be $\langle r \rangle = a = 5.29 \times 10^{-9}$ cm.

   (b) What is the radius of the $n = 10$ orbitals in Hydrogen?

   Solution: The answer to this question depends on what is meant by "radius" (all reasonable answers will get full credit). Two possible answers are the following. The radial wavefunction for an arbitrary orbital is of the form:

   $$R_{nl}(r) = r\rho^{l-1}e^{-\rho}v(\rho), \tag{2.4}$$

   where $\rho = \frac{r}{na}$. From the form of this wavefunction, we can see that the exponential term starts to cut the wavefunction at $\rho \sim 1$, or $r \sim na$. Thus, we can say that the "radius" of the n-th orbital is given by the factor na. Perhaps a better and less ambiguous way is to define "radius" as the expectation value of r. The result is:

   $$\langle r \rangle = a \cdot n^2 \{\frac{3}{2} - \frac{l(l+1)}{n^2}\}. \tag{2.5}$$

   Thus $\langle r \rangle \approx a \cdot n^2 \cdot \frac{3}{2} \sim 10^{-6}$ cm.

   A fast way to obtain this scaling is to say that the energy of the electron ($E = Ry/n^2 \sim e^2/\langle r \rangle$), from which one sees that $\langle r \rangle$ scales as n^2.

   (c) What is the First Bohr radius in hydrogenic nitrogen (an ion consisting of a nitrogen nucleus and just one electron)? What is the ionization energy (in eV) for the ground state of this ion (i.e. the minimum energy we need to supply to strip the ion of its last remaining electron)?

Solution: The first Bohr radius in hydrogenic nitrogen can be found by just noting that the only modification to the hydrogen problem from having a nitrogen core is that we need to modify all terms involving $e^2 \longrightarrow Ze^2$. In particular, this means that the Bohr radius must be modified as

$$a = \frac{\hbar^2}{me^2} \longrightarrow a/Z = \frac{\hbar^2}{Zme^2} = a_0/7 = 7.56 \times 10^{-10} \text{ cm}, \tag{2.6}$$

and the ionization energy must be modified as

$$-E_1 = \frac{me^4}{2\hbar^2} \longrightarrow Z^2 E_1 = \frac{mZ^2 e^4}{2\hbar^2} = 49 \cdot 13.6 \text{ eV} = 666 \text{ eV} \tag{2.7}$$

3. List all possible states with n=3 in hydrogen. Label each state with a complete set of quantum numbers. Which of the states are exactly degenerate in the absence of external fields?

   Solution: There are nine

$$\sum_{l=0}^{n-1} (2l+1) = n^2 = 3^2 \tag{2.8}$$

   possible states in spinless hydrogen with $n = 3$. In the absence of external fields (and neglecting any higher order corrections or spin effects), they are all degenerate, since the energy of states in the Hydrogen atom only depend on n, $E_{nlm} = -13.6$ eV $\cdot \frac{1}{n^2}$. These nine states are found by considering the allowed values of l and m for $n = 3$. The allowed values of l are from 0 to $n-1$, so we have $l = 0, 1, 2$. For each allowed value of l we have $2l + 1$ allowed values of m. Thus, we find that the allowed states are:

$$
\begin{aligned}
l &= 0 & m &= 0 \\
l &= 1 & m &= -1 \\
l &= 1 & m &= 0 \\
l &= 1 & m &= 1 \\
l &= 2 & m &= -2 \\
l &= 2 & m &= -1 \\
l &= 2 & m &= 0 \\
l &= 2 & m &= 1 \\
l &= 2 & m &= 2.
\end{aligned}
$$

   Hydrogen is a special case of a centro-symmetric potential where there is "accidental" degeneracy between levels of different l and same n. Various interactions within the atom (e.g. the LS-interaction) leading to deviations from a pure Coulomb potential can lift degeneracies between different l states. However, in the absence of external fields, states with same n, l but different m remain degenerate. This is a consequence of isotropy of space, or in other words, the absence of preferred directions.

4. For a state with given l and m, find $\langle L_x \rangle$, $\langle L_y \rangle$, $\langle L_x^2 \rangle$, and $\langle L_y^2 \rangle$.

   Solution: We use the properties of the angular momentum ladder operators and the states $|lm\rangle$ to do this computation. In particular, we use the following facts about the ladder operators,

$$L_\pm = L_x \pm iL_y \tag{2.9}$$

$$L_x = \frac{1}{2}(L_+ + L_-) \tag{2.10}$$

$$L_y = \frac{1}{2i}(L_+ - L_-) \tag{2.11}$$

$$L_\pm |lm\rangle = \hbar\sqrt{l(l+1) - m(m\pm 1)}|lm\pm 1\rangle, \tag{2.12}$$

and the following fundamental property of the states $|lm\rangle$

$$\langle l'm'|lm\rangle = \delta_{ll'}\delta_{mm'}. \tag{2.13}$$

Now, using the ladder operator relations in Eqns. (2.12) and the orthonormality relation (2.13), first consider:

$$\langle lm|L_{\pm}|lm\rangle = \hbar\sqrt{l(l+1) - m(m \pm 1)}\langle lm|lm \pm 1\rangle = 0. \tag{2.14}$$

Using this fact we clearly see that:

$$\langle L_x \rangle = \langle lm|\frac{1}{2}(L_+ + L_-)|lm\rangle = 0 \tag{2.15}$$

$$\langle L_y \rangle = \langle lm|\frac{1}{2i}(L_+ - L_-)|lm\rangle = 0. \tag{2.16}$$

In fact, from the orthonormality relation, Eqn. (2.13), and the effect of the ladder operators, Eqn. (2.12), we see that the only terms which contribute to these expectation values are those with an equal number of raising and lowering operators (which just give me some multiple of my original state back again). So in particular, we have

$$\langle L_x^2 \rangle = \langle lm|\frac{1}{4}(L_+ + L_-)(L_+ + L_-)|lm\rangle = \langle lm|\frac{1}{4}(L_-L_+ + L_+L_-)|lm\rangle. \tag{2.17}$$

Now, from my review notes, we also note that:

$$L_{\pm}L_{\mp} = (L_x \pm iL_y)(L_x \mp iL_y) = L_x^2 + L_y^2 \mp i[L_x, L_y] = \mathbf{L^2} - L_z^2 \pm \hbar L_z. \tag{2.18}$$

Thus, we find that:

$$\langle L_x^2 \rangle = \langle lm|\frac{1}{4}(L_-L_+ + L_+L_-)|lm\rangle = \frac{1}{2}\langle lm|\mathbf{L^2} - L_z^2|lm\rangle$$
$$= \frac{1}{2}\langle lm|\mathbf{L^2} - L_z^2|lm\rangle = \frac{\hbar^2}{2}\left(l(l+1) - m^2\right). \tag{2.19}$$

We don't have to do any more computations as we have that:

$$\langle L_y^2 \rangle = -\langle lm|\frac{1}{4}(L_+ - L_-)(L_+ - L_-)|lm\rangle = \langle lm|\frac{1}{4}(L_-L_+ + L_+L_-)|lm\rangle$$
$$= \langle L_x^2 \rangle = \frac{\hbar^2}{2}\left(l(l+1) - m^2\right). \tag{2.20}$$

Another way to solve this problem is just to use the vector model, in which the angular momentum vector has length $\hbar \cdot \sqrt{l(l+1)}$. From the Pythagoras theorem, we have $\hbar^2 \cdot l(l+1) = (\hbar \cdot m)^2 + L_x^2 + L_y^2$. Averaging this expression and taking into account that from symmetry $\langle L_y^2 \rangle = \langle L_x^2 \rangle$, we obtain the same answer as above.