I. Basics + Logistics

① According to the Online Schedule, Final is in 101 LSA at 8:00 AM, Tuesday, December 19th!

② Basic Info - Final is 5-6 problems, most involve little computation - mostly conceptual, physical. It covers wave mechanics, uncertainty principle, one-dimensional problems (1D-SHO, infinite square well), angular momentum / Hydrogen Atom, + perturbation theory. I'll cover the first three topics today + the last two along with solutions to the review problems next Sunday.

③ There will certainly be a cheat sheet allowed.

II. Fundamental Principles of QM I: Schrödinger's Eqn

Time dependent Schrödinger's Equation: $i\hbar \frac{\partial \Psi}{\partial t} = H\Psi$ describes the time evolution of any state/wave-function which describes some physical system. Two very important facts!

① This equation is linear! Thus, if Ψ and χ are both solutions, so is: $a\Psi + b\chi$ for $a, b \in \mathbb{C}$. In particular, suppose I have a set of solutions $\{\Psi_n\}$ which, at time $t=0$, form a complete + orthonormal set of functions, (i.e. $\int \Psi_n^*(x,t=0)\Psi_m(x,t=0)dx = \delta_{nm}$ and $\exists \{c_n\}$ s.t. for any $\Psi(x)$, $\Psi(x) = \sum_{n=0}^{\infty} c_n \Psi_n(x,t=0)$), then for any initial condition, $\Psi(x,t=0)$, we can find its time evolution:

This $\Rightarrow$ is why the space of states is a vector space. This is equiv. to the superposition principle!

(a) Expand $\Psi(x,t=0)$ in terms of the Ψ_n:

$$\Psi(x,t=0) = \sum_{n=0}^{\infty} c_n \Psi_n(x,t=0), \text{ where } c_n = \int \Psi_n^*(x,t=0)\Psi(x,t=0)dx$$

(b) The time dependent solution with initial condition $\Psi(x,t=0)$ is then just:

$$\Psi(x,t) = \sum_{n=0}^{\infty} c_n \Psi_n(x,t) \Rightarrow \text{so finding a set } \{\Psi_n\} \text{ allows us to solve Schr. Eqn completely}$$

② Special Case: Suppose H is time independent. Then, we can use separation of variables + the time independent Schrödinger equation to find and solve for a particular set of solutions $\{\Psi_n\}$ which are complete + orthonormal, and by ① above, we've solved our problem! Note: This will not work if H is time dependent!!

In particular, let's work this out. We make the ansatz that $\Psi_n(x,t) = \chi_n(t)\psi_n(x) \Rightarrow H\psi_n(x) = E_n\psi_n(x)$, $\chi_n(t) = e^{-iE_nt/\hbar}$

Thus, we see that this complete set $\{\Psi_n(x,t) = \Psi_n(x)e^{-iE_nt/\hbar}\}$ is just the set of normalized eigenfunctions of the Hamiltonian! Thus, we find: Given $\Psi(x,t=0)$, H, we should:

ⓐ Solve $H\Psi_n(x) = E_n\Psi_n(x)$ to obtain $\{\Psi_n(x,t) = \Psi_n(x)e^{-iE_nt/\hbar}\}$ a complete set of orthonormal eigenfunctions of H.

ⓑ Expand $\Psi(x,t=0) = \sum_{n=0}^{\infty} c_n\Psi_n(x)$ $\quad c_n = \int \Psi_n^*(x)\Psi(x,t=0)dx$

ⓒ We're done! Our solution is: $\Psi(x,t) = \sum_{n=0} c_n\Psi_n(x)e^{-iE_nt/\hbar}$

III. Fundamentals II: Measurement, Observables, Uncertainty.

Observables in QM are represented by Hermitian operators $\mathcal{O}$, acting on the vector space (∞-dim'l Hilbert space) of states of the physical system. For example:

X, $H = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) = \frac{p^2}{2m} + V(x)$, $\quad P_x = -i\hbar\frac{\partial}{\partial x}$, $\quad \vec{L} = \vec{r}\times\vec{p}$

$L_z = (xp_y - yp_x) = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -i\hbar\frac{\partial}{\partial\phi}$

The fact that they are Hermitian means that $\mathcal{O}^\dagger = \mathcal{O}$, where $\mathcal{O}^\dagger$ is defined by the condition that:

$$\langle\mathcal{O}\Psi|\chi\rangle = \int(\mathcal{O}\Psi)^*\chi\,dx \overset{\text{def}}{\equiv} \int\Psi^*(\mathcal{O}^\dagger\chi)dx = \langle\Psi|\mathcal{O}^\dagger\chi\rangle$$

[I've used bra-ket notation here: $\chi(x) \longmapsto |\chi\rangle$ while $\int\Psi^*(x)\cdot dx \longmapsto \langle\Psi|$, so $\langle\Psi|\chi\rangle = \int\Psi^*(x)\chi(x)dx$!]

Given an observable $\mathcal{O}$, we'd like to know what values it can take in a given state, how probable each of them are and the average value of $\mathcal{O}$ for any given state. To do this, we do the following:

① Find a complete set of eigenvectors of $\mathcal{O}$, i.e. $\{\phi_n(x)\}$ (or $\{|\phi_n\rangle\}$. s.t. $\mathcal{O}\phi_n(x) = \lambda_n\phi_n(x)$. The set $\{\lambda_n\}$ is the set of all possible outcomes of a measurement of $\mathcal{O}$ (for any possible ~~initial~~ state)!

② For a given state $\Psi(x,t)$, (or $|\Psi\rangle$), the probability of measuring the value λ_n for the observable $\mathcal{O}$ is given by:

$$P(\lambda_n) = |\langle\phi_n|\Psi\rangle|^2 = \left|\int\phi_n^*(x)\Psi(x,t)dx\right|^2$$

③ The average value (expectation value) of $\mathcal{O}$ in a state Ψ is just:

$$\langle\Psi|\mathcal{O}|\Psi\rangle = \int\Psi^*(x,t)\mathcal{O}\Psi(x,t)dx = \sum_{n=0}^{\infty}P(\lambda_n)\lambda_n = \langle\mathcal{O}\rangle$$

④ The uncertainty in $\mathcal{O}$ is given by:

$$\sqrt{\langle(\mathcal{O}-\langle\mathcal{O}\rangle)^2\rangle} = \sqrt{\langle\mathcal{O}^2\rangle - \langle\mathcal{O}\rangle^2} = (\Delta\mathcal{O})$$

IV. Fundamentals III: Wave Packets, Dispersion, Heisenberg Uncertainty.
It is worthwhile reviewing Budker's HW3, problems 1+2.
1D Free particle: $V(x)=0$ $\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}=E\psi$, $k\equiv\frac{\sqrt{2mE}}{\hbar}$
The solutions are $\psi(x)=Ae^{ikx}+Be^{-ikx}$, we get time dep. by adding $e^{-iEt/\hbar}$,

$$\Rightarrow \psi(x,t)=A\underbrace{\exp\left(ik\left(x-\tfrac{\hbar k}{2m}t\right)\right)}_{\text{right moving wave}}+B\underbrace{\exp\left(-ik\left(x+\tfrac{\hbar k}{2m}t\right)\right)}_{\text{left moving wave}}$$

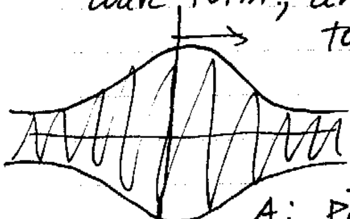
Apparently, these are moving at the velocity $V_p=\pm\frac{\hbar k}{2m}$ (phase velocity)
But this makes no sense! Their kinetic energy is:

$$E=K.E.=\frac{\hbar^2k^2}{2m}=\frac{1}{2}mv^2\Rightarrow v=\frac{\hbar k}{m}\neq\hbar k/2m\ ?!?$$

What's going on? First, note that this wave f^n is NOT normalizable
$\Rightarrow\int\psi^*\psi=\infty$! Thus it isn't a physic. realizable state!
We need to construct a wave packet, a sum of these ~~eigenstates~~!

$$\psi(x,t)=\frac{1}{\sqrt{2\pi}}\int\phi(k)e^{i(kx-\frac{\hbar k^2}{2m}t)}dk$$

which is normalizable for an appropriate choice of $\phi(k)$.
It turns out that a particle is represented by a localized wave form; and the speed of the particle should be related to the speed of the motion of the wave form itself, not of any particular $e^{i(kx-\omega t)}$ term in this superposition.



Q: What is the speed of the wave form?
A: Principle of Stationary phase tells us that an integral of the form $\int\phi(k)\exp(i\beta(k))dk$ is dominated by the region of the integral where β is stationary (either minimum or maximum). Why? If β is near a min or max, $\frac{\partial\beta}{\partial k}=0$, so β is nearly constant. Thus, the ~~integral doesn't~~ integrand isn't oscillating very much (as it would any where that β is varying), and won't cancel out to 0. In our case, we have:

$$\psi(x,t)=\frac{1}{\sqrt{2\pi}}\int\phi(k)\exp(ikx-\omega(k)t))dk\qquad \omega(k)=\frac{\hbar^2k^2}{2m}$$

$$\Rightarrow \text{dominated by } \frac{\partial}{\partial k}(kx-\omega(k)t)=0\Rightarrow x-\frac{\partial\omega}{\partial k}t=0\Rightarrow\frac{x}{t}=\frac{\partial\omega}{\partial k}$$

Thus, $\psi(x,t)$ is dominated by values where $\frac{x}{t}=\frac{\partial\omega}{\partial k}\equiv V_g$, and thus looks like a wave form travelling at group velocity $V_g=\frac{\partial\omega}{\partial k}$, rather than the phase velocity $V_p=\frac{\omega}{k}$.
In particular, in QM, we have for a free particle $\omega(k)=\frac{\hbar k^2}{2m}$

$\Rightarrow V_g=\frac{\hbar k}{m}$, $V_p=\frac{\hbar k}{2m}$, so a wave packet moves at the speed we'd expect!

Q: What are the group + phase velocities for a relativistic particle of mass m?

A: In this case $E^2 = p^2c^2 + m^2c^4 = \hbar^2 k^2 c^2 + m^2 c^4$

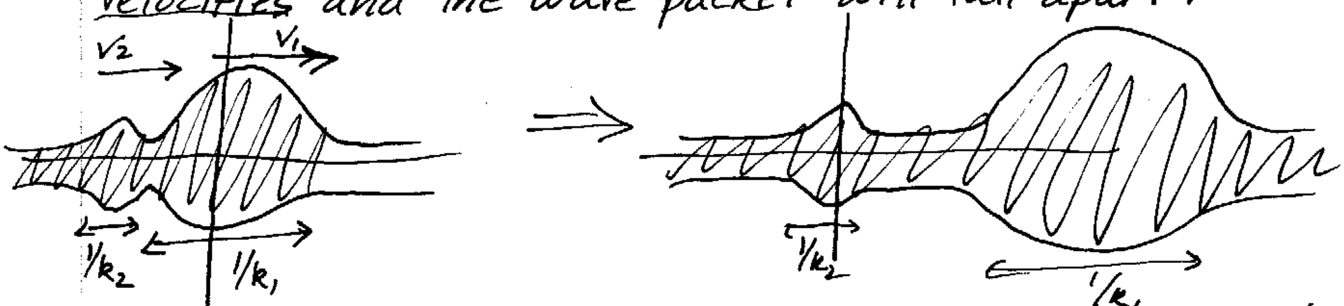
So, using the relations $E = \hbar\omega$, $p = \hbar k$, we have:

$$\hbar^2 \omega(k)^2 = \hbar^2 k^2 c^2 + m^2 c^4 \implies \omega(k) = c\sqrt{k^2 + \frac{m^2c^2}{\hbar^2}}$$

Thus, $V_p = \frac{\omega(k)}{k} = \frac{c\sqrt{k^2 + (mc/\hbar)^2}}{k}$ while $V_g = \frac{c}{2}\frac{2k}{\sqrt{k^2 + (mc/\hbar)^2}} = \frac{ck}{\sqrt{k^2 + (mc/\hbar)^2}}$

Q: Does a relativistic wave packet generally disperse? What about an ultra-relativistic wave packet $p \gg mc$?

A: The dispersion of a wave packet is related to the question of whether the group velocity is constant as a function of k! If the group velocities of wave-forms ~~with of~~ with different values of k are different, then features with size $\sim 1/k_1$ and those with size $1/k_2$ will move at different velocities and the wave packet will fall apart!



Thus, we require the group velocity, V_g to be constant as a function of k!

No Dispersion $\iff \frac{\partial V_g}{\partial k} = 0 \iff \frac{\partial^2 \omega}{\partial k^2} = 0$!

Clearly, a relativistic wave packet disperses as $\frac{\partial^2 \omega}{\partial k^2} = \frac{c}{\sqrt{k^2 + (mc/\hbar)^2}} - \frac{ck^2}{(k^2 + (mc/\hbar)^2)^{3/2}} = (mc/\hbar)^2 \left(k^2 + (mc/\hbar)^2\right)^{3/2}$

However, in the ultra relativistic limit, $k^2 \gg (mc/\hbar)^2$, $\frac{\partial^2 \omega}{\partial k^2} \to 0$, so we expect little or no dispersion!