ATOMIC PHYSICS
An Exploration Through Problems and Solutions

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To our teachers
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CONTENTS

Preface xi

Notation xiii

1 Atomic structure 1
   1.1 Ground state of phosphorus 1
   1.2 Exchange interaction 7
   1.3 Spin-orbit interaction 10
   1.4 Hyperfine structure and Zeeman effect in hydrogen 13
   1.5 Hydrogenic ions 18
   1.6 Geonium 22
   1.7 The Thomas-Fermi model (T) 31
   1.8 Electrons in a shell 34
   1.9 Isotope shifts and the King plot 38
   1.10 Crude model of a negative ion 41
   1.11 Hyperfine-interaction-induced mixing of states of different J 43
   1.12 Electron density inside the nucleus (T) 46
   1.13 Parity nonconservation in atoms 52
   1.14 Parity nonconservation in anti-atoms 62
   1.15 The anapole moment (T) 65

2 Atoms in external fields 74
   2.1 Electric polarizability of the hydrogen ground state 74
   2.2 Polarizabilities for highly excited atomic states 77
   2.3 Using Stark shifts to measure electric fields 78
   2.4 Larmor precession frequencies for alkali atoms 80
   2.5 Magnetic field inside a magnetized sphere 83
   2.6 Classical model of magnetic resonance 83
   2.7 Energy level shifts due to oscillating fields (T) 88
   2.8 Spin relaxation due to magnetic field inhomogeneity 100
   2.9 The $\vec{E} \times \vec{v}$ effect in vapor cells 105
   2.10 Field ionization of hydrogenic ions 108
   2.11 Electric-field shifts of magnetically split Zeeman sublevels 108
   2.12 Geometric (Berry’s) phase 110
3 Interaction of atoms with light

3.1 Two-level system under periodic perturbation (T) 115
3.2 Quantization of the electromagnetic field (T) 122
3.3 Emission of light by atoms (T) 128
3.4 Absorption of light by atoms 138
3.5 Resonant absorption cross-section 141
3.6 Absorption cross-section for a Doppler-broadened line 143
3.7 Saturation parameters (T) 145
3.8 Angular distribution and polarization of atomic fluorescence 152
3.9 Change in absorption due to optical pumping 156
3.10 Optical pumping and the density matrix 162
3.11 Cascade decay 166
3.12 Coherent laser excitation 168
3.13 Transit-time broadening 170
3.14 A quiz on fluorescence and light scattering 172
3.15 Two-photon transition probability 177
3.16 Vanishing Raman scattering 179
3.17 Excitation of atoms by off-resonant laser pulses 180
3.18 Hyperfine-interaction-induced magnetic dipole (M1) transitions 184

4 Interaction of light with atoms in external fields 187

4.1 Resonant Faraday rotation 187
4.2 Kerr effect in an atomic medium 191
4.3 The Hanle effect 197
4.4 Electric-field-induced decay of the hydrogen $2^2S_{1/2}$ state 200
4.5 Stark-induced transitions (T) 202
4.6 Magnetic deflection of light 208
4.7 Classical model of an optical-pumping magnetometer 213
4.8 Searches for permanent electric dipole moments (T) 217

5 Atomic collisions 230

5.1 Collisions in a buffer gas 230
5.2 Spectral line broadening due to phase diffusion 231
5.3 Dicke narrowing 234
5.4 Basic concepts in spin exchange 238
5.5 The spin-temperature limit 242
5.6 Electron-randomization collisions 244
5.7 Larmor precession under conditions of rapid spin exchange 245
5.8 Penning ionization of metastable helium atoms 247
CONTENTS

6 Cold atoms

6.1 Laser cooling: basic ideas (T) 251
6.2 Magneto-optical traps 258
6.3 Zeeman slower 262
6.4 Bose-Einstein condensation (T) 267
6.5 Bose-Einstein condensation from an optical lattice 278
6.6 Cavity cooling 280
6.7 Cavity cooling for many particles; stochastic cooling 285
6.8 Matter-wave vs. optical Sagnac gyroscopes 287
6.9 Fermi energy for a harmonic trap 291

7 Molecules

7.1 Amplitude of molecular vibrations 294
7.2 Vibrational constants for the Morse potential 295
7.3 Centrifugal distortion 297
7.4 Relative densities of atoms and molecules in a vapor 300
7.5 Isotope shifts in molecular transitions 305
7.6 Electric dipole moments of polar molecules 310
7.7 Scalar coupling of nuclear spins in molecules 314

8 Experimental methods

8.1 Reflection of light from a moving mirror 319
8.2 Laser heating of a small particle 321
8.3 Spectrum of frequency-modulated light 324
8.4 Frequency doubling of modulated light 326
8.5 Ring-down of a detuned cavity 328
8.6 Transmission through a light guide 329
8.7 Quantum fluctuations in light fields 330
8.8 Noise of a beamsplitter 334
8.9 Photon shot noise in polarimetry 336
8.10 Light-polarization control with a variable retarder 338
8.11 Pile-up in photon counting 341
8.12 Photons per mode in a laser beam 342
8.13 Tuning dye lasers 344
8.14 Femtosecond laser pulses and frequency combs 347
8.15 Magnetic field fluctuations due to random thermal currents 351

9 Miscellaneous topics

9.1 Precession of a compass needle? 355
9.2 Ultracold neutron polarizer 357
<table>
<thead>
<tr>
<th>CONTENTS</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.3 Exponentially growing/decaying harmonic field</td>
<td>358</td>
</tr>
<tr>
<td>9.4 The magic angle</td>
<td>360</td>
</tr>
<tr>
<td>9.5 Understanding a Clebsch-Gordan coefficient selection rule</td>
<td>366</td>
</tr>
<tr>
<td>9.6 The Kapitsa pendulum</td>
<td>368</td>
</tr>
<tr>
<td>9.7 Visualization of atomic polarization</td>
<td>371</td>
</tr>
<tr>
<td>A Units, conversion factors, and typical values</td>
<td>379</td>
</tr>
<tr>
<td>B Reference data for hydrogen and alkali atoms</td>
<td>385</td>
</tr>
<tr>
<td>C Spectroscopic notation for atoms and diatomic molecules</td>
<td>387</td>
</tr>
<tr>
<td>D Description of polarization states of light</td>
<td>390</td>
</tr>
<tr>
<td>D.1 The Stokes parameters</td>
<td>390</td>
</tr>
<tr>
<td>D.2 The Jones calculus</td>
<td>391</td>
</tr>
<tr>
<td>E Euler angles and rotation matrices</td>
<td>393</td>
</tr>
<tr>
<td>F The Wigner-Eckart theorem and irreducible tensors</td>
<td>395</td>
</tr>
<tr>
<td>F.1 Wigner-Eckart theorem</td>
<td>395</td>
</tr>
<tr>
<td>F.2 Irreducible tensors</td>
<td>401</td>
</tr>
<tr>
<td>G The density matrix</td>
<td>403</td>
</tr>
<tr>
<td>G.1 Connection between the density matrix and the wavefunction</td>
<td>403</td>
</tr>
<tr>
<td>G.2 Ensemble-averaged density matrix</td>
<td>406</td>
</tr>
<tr>
<td>G.3 Time evolution of the density matrix: the Liouville equation</td>
<td>408</td>
</tr>
<tr>
<td>G.4 Atomic polarization moments</td>
<td>410</td>
</tr>
<tr>
<td>H Elements of the Feynman diagram technique</td>
<td>415</td>
</tr>
<tr>
<td>Bibliography</td>
<td>418</td>
</tr>
<tr>
<td>Index</td>
<td>434</td>
</tr>
</tbody>
</table>
PREFACE

We have found that usually the best way to learn something new is to ask concrete questions and try to work out the answers. Often some of the simplest questions have surprising and unexpected answers, and some seemingly complex problems can be solved in a simple way. In this book we have collected some of these problems and our solutions to them. The book encompasses many issues we faced as we ourselves made the transition from undergraduate students to practicing experimental atomic physicists and instructors. However, the text is not intended to be comprehensive, but rather addresses various aspects of atomic physics which we have found interesting and important.

In the course of doing atomic physics, we always find ourselves crossing boundaries into other subfields; the selection of problems reflects this gray area. It also reflects our specific interests, with several problems about symmetry violation, etc. that would not appear in more “standard” textbooks. It is our philosophy that working on specific problems usually helps with understanding of more general issues, and indeed may be the most useful way to really learn anything. It is our hope that some selection of the broad range of problems given here will pique the interest of any reader, and thus initiate this process.

Where possible, we try to emphasize approximation methods, dimensional considerations, limiting cases, and symmetry arguments as opposed to formal mathematics. We often appeal to pictures, tables, and graphs. This problem-solving approach is aimed at developing intuition about physical principles and fosters the important ability to perform “back-of-the-envelope” calculations. These are the tools we find most useful when trying to solve the types of problems we commonly encounter in the laboratory. Of course, on occasion a formal mathematical approach (as painful as it could be) can lead to important insights. Generally, in order to deeply understand various aspects of physics, it is good to have both an intuitive picture as well as the appropriate mathematical tools.

This book is intended for advanced undergraduates and beginning graduate students interested in atomic, molecular and optical physics, and we assume that readers possess basic knowledge of quantum mechanics [at the level of Griffiths (1995), Bransden and Joachain (1989), or similar texts], electrodynamics [at the level of Griffiths (1999), Purcell (1985), or similar texts], and thermodynamics [at the level of Reif (1965), Kittel and Kroemer (1980), or similar texts]. However, we hope that many of the problems will also be of interest to professional scientists.
In physics, there continues to be a raging debate over what is the best system of units to use and whether or not units should be standardized. We feel that the choice of units is a personal one, especially since converting between different systems is relatively straightforward. That said, in this book we have a tendency to use CGS units, since we find them most convenient (especially in problems involving electromagnetism). We also set $\hbar = 1$ when it is convenient to do so and measure energies in frequency units, as is common practice in atomic physics (since energy measurements are typically performed by measuring frequencies).

Each problem in the book is intended to stand on its own. If there is a particular subject in atomic physics that one is interested in learning about, there may be a problem about it in this book. We envision the reader turning right to that page and starting to try to figure it out. Hopefully, at the end of this exercise, one will have gained some familiarity with the topic, enabling her or him to understand more advanced, specialized literature on the subject, or go straight to the lab and get to work!

In the introduction to most problems there is a brief discussion of the relevance of the topic to modern atomic physics with references to research literature on the subject. The cited references are not intended to be comprehensive, but merely provide a starting point in a search for more information about the subject of the problem. We apologize in advance to the innumerable scientists whose important contributions are not mentioned. Also, for a few problems, especially in subfields of atomic physics dear to our hearts, there are some historical remarks. Of course, there is a great deal of history surrounding almost all of the topics covered in this book, and we could not tell all of it. Nonetheless, we thought a few, not widely known stories might be enjoyable.

Some of the problems are written as tutorials on various subjects in atomic physics [they are marked with a (T)]. In such problems, there are a series of short questions that are intended to guide the reader through some important material. Hopefully the reader will find this more entertaining and interactive than just reading the explanation straight through.

We hope you enjoy reading and using the book as much as we have enjoyed writing it!

D. B.
D. F. K.
D. P. D.

Berkeley, California
May 2003
## NOTATION

The following is a table of symbols commonly used throughout the book, their meaning, and their value where appropriate. In most locations, we remind the reader of the meaning of the symbols when they appear. Also see Appendix A for practical units, conversion factors, and typical values of various parameters.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
<th>Value</th>
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<tbody>
<tr>
<td>$m, m_e$</td>
<td>electron mass</td>
<td>$9.1085 \times 10^{-28}$ g, $0.511$ MeV/c$^2$</td>
</tr>
<tr>
<td>$m_p$</td>
<td>proton mass</td>
<td>$1.6726 \times 10^{-24}$ g, $938.28$ MeV/c$^2$</td>
</tr>
<tr>
<td>$m_n$</td>
<td>neutron mass</td>
<td>$1.6750 \times 10^{-24}$ g, $939.57$ MeV/c$^2$</td>
</tr>
<tr>
<td>$m_n - m_p$</td>
<td>difference between nucleon masses</td>
<td>$1.293$ MeV/c$^2$</td>
</tr>
<tr>
<td>$e$</td>
<td>electron charge magnitude</td>
<td>$4.8029 \times 10^{-10}$ esu</td>
</tr>
<tr>
<td>$h$</td>
<td>Planck’s constant</td>
<td>$6.6252 \times 10^{-27}$ erg $\cdot$ s</td>
</tr>
<tr>
<td>$h = h/(2\pi)$</td>
<td></td>
<td>$1.0544 \times 10^{-27}$ erg $\cdot$ s</td>
</tr>
<tr>
<td>$\alpha = e^2/(hc)$</td>
<td>fine structure constant</td>
<td>$1/137.036$</td>
</tr>
<tr>
<td>$a_0 = h^2/(me^2)$</td>
<td>Bohr radius</td>
<td>$5.292 \times 10^{-9}$ cm</td>
</tr>
<tr>
<td>$\mu_0 = eh/(2mc)$</td>
<td>Bohr magneton</td>
<td>$0.93 \times 10^{-20}$ erg/G, $1.40$ MHz/G</td>
</tr>
<tr>
<td>$\mu_N = eh/(2m_pc)$</td>
<td>nuclear magneton</td>
<td>$5.06 \times 10^{-24}$ erg/G, $762$ Hz/G</td>
</tr>
<tr>
<td>$R_\infty = me^4/(4\pi\hbar^3c)$</td>
<td>Rydberg constant</td>
<td>$109,737$ cm$^{-1}$</td>
</tr>
<tr>
<td>$k_B$</td>
<td>Boltzmann’s constant</td>
<td>$1.38066 \times 10^{-16}$ erg/K, $8.61735 \times 10^{-5}$ eV/K</td>
</tr>
<tr>
<td>$L, l$</td>
<td>orbital angular momentum (total, individual particle)</td>
<td>units of $\hbar$</td>
</tr>
<tr>
<td>$S, s$</td>
<td>electron spin</td>
<td>units of $\hbar$</td>
</tr>
<tr>
<td>$J, j$</td>
<td>total electronic angular momentum</td>
<td>units of $\hbar$</td>
</tr>
<tr>
<td>$I$</td>
<td>nuclear spin</td>
<td>units of $\hbar$</td>
</tr>
<tr>
<td>$F$</td>
<td>total atomic angular momentum</td>
<td>units of $\hbar$</td>
</tr>
</tbody>
</table>
NOTATION

When we deal with spin-1/2 systems, we will commonly employ the notation $|\pm\rangle$ and $|\mp\rangle$ to denote the spin up ($m = +1/2$) and spin down ($m = -1/2$) states, respectively. Here $m$ is the projection of the spin along the quantization axis.

The ubiquitous Clebsch-Gordan coefficients\footnote{The Clebsch-Gordan coefficients are also referred to as vector-coupling coefficients, vector-addition coefficients, and Wigner coefficients in the literature.} describe the connection between the coupled basis $|J, M\rangle$ and the uncoupled basis $|J_1, M_1\rangle|J_2, M_2\rangle$ (where $J, J_1, J_2$ are angular momenta and $M, M_1, M_2$ are the projections of the respective angular momenta on the quantization axis):

\[
|J, M\rangle = \sum_{M_1, M_2} C(J_1, J_2, J; M_1, M_2, M)|J_1, M_1\rangle|J_2, M_2\rangle \tag{2.1}
\]

\[
|J_1, M_1\rangle|J_2, M_2\rangle = \sum_{J, M} C(J_1, J_2, J; M_1, M_2, M)|J, M\rangle. \tag{2.2}
\]

In the text we consistently use the notation:

\[
C(J_1, J_2, J; M_1, M_2, M) \equiv \langle J_1, M_1, J_2, M_2|J, M\rangle, \tag{2.3}
\]