Physics H7A (Budker) Fall 2009
Midterm 1 Review
October 6-7, 2009

1. Lagrange points and pictures Consider two massive objects, say the Earth and the Moon. Let the distance between them be R. If these objects were at fixed locations (i.e. not rotating about each other), there would be one location where a small satellite of mass m could remain fixed. Draw a picture of the Earth-Moon system and label approximately where this points is.

Now consider the case where the Moon is orbiting the Earth. (So long as the Earth is significantly heavier than the Moon, which it is, we can assume the Earth is fixed and the Moon rotates about the Earth's center.) There are now *five* points in the plane of rotation where an object can remain relatively at rest in the rotating frame of the Moon about the Earth. These are known as the Lagrangian points. Find (very) roughly where they are located and explain, using a free body diagram, why the mass m would be stable there.

See the wikipedia page on "Lagragian point" for a thorough discussion and a very pretty picture of the Earth/Sun system. (The Earth/Moon system is similar enough.)

The basic idea is that in the rotating frame, there are three forces acting at a given location: the gravitational attraction of the Earth, that of the Moon, and the centrifugal force. Because the Earth is not infinitely massive, the gravitational force and centrifugal force are not exactly in the same direction. This is because the center of rotation of the system is *not* at the Earth's center but at the center of mass of the Earth and the Moon. So in addition to the obvious points 1, 2, and 3 in the wikipedia picture, there are the two other points 4 and 5 where the slight angle between the centrifugal force and the Earth's gravitational force allow the cancellation of the Moon's force.

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2. Multi-stage or single-stage rocket A rocket of (empty) mass m_0 is designed with two identical boosters, which when ignited burn their fuel at a constant rate of $b = dm/dt$ for a time T. (So the initial mass of the rocket+fuel is $m_0 + 2bT$.) In the absence of gravity, is it better for the rocket to ignite its boosters sequentially or simultaneously? What about if the boosters have an (empty) mass m_b and we're going to jettison the boosters once each is empty? What about in the presence of gravity?

Rederiving the rocket formula is straightforward. In the presence of gravity, one finds

$$v(t) = v(0) + u \log\left(\frac{m(0)}{m(t)}\right) - gt$$

where $m(0)$ is the initial mass and $m(t)$ is the mass at time t.

In the absence of gravity and if the boosters are massless, there is no advantage to doing it one way versus the other. The reason is that firing both boosters simultaneously gives

$$v_{sim} = u \log\left(\frac{m_0 + 2bT}{m_0}\right)$$

If the boosters are fired sequentially, one finds

$$v_{seq} = u \log\log\left(\frac{m_0 + bT}{m_0}\right) + u \log\log\left(\frac{m_0 + 2bT}{m_0 + bT}\right) = u \log\left(\frac{m_0 + 2bT}{m_0}\right)$$

If the boosters have mass m_b and are jettisoned, we find for simultaneous firing

$$v_{sim} = u \log\left(\frac{m_0 + 2m_b + 2bT}{m_0 + 2m_b}\right)$$

For sequential firing,

$$v_{seq} = u \log \left(\frac{m_0 + 2m_b + 2bT}{m_0 + 2m_b + bT} \right) + u \log \left(\frac{m_0 + m_b + bT}{m_0 + m_b} \right)$$

It is somewhat clear that sequential should be faster (we don't waste time accelerating the second booster)

$$\begin{aligned} v_{seq} - v_{sim} &= u \log \left(\frac{m_0 + 2m_b}{m_0 + 2m_b + bT} \right) + u \log \left(\frac{m_0 + m_b + bT}{m_0 + m_b} \right) \\ &= u \log \left(\frac{m_0 + 2m_b}{m_0 + m_b} \right) + u \log \left(\frac{m_0 + m_b + bT}{m_0 + 2m_b + bT} \right) \end{aligned}$$

where we've rearranged the logarithm arguments. This is positive. To see why, note that the first log argument is a number greater than 1 and so its log is positive. The second log argument is the reversed fraction, but with the quantity bT added to both numerator and denominator, which makes the fraction closer to one, and so its log, while negative, has a smaller magnitude than the first. Thus the sum is positive.

When gravity is added, things change a bit. For the first case of massless boosters, we must subtract gT from v_{sim} but $2gT$ from v_{seq}, so v_{sim} is better. For massive boosters, we find a difference of

$$v_{seq} - v_{sim} = u \log \left(\frac{m_0 + 2m_b}{m_0 + m_b} \right) + u \log \left(\frac{m_0 + m_b + bT}{m_0 + 2m_b + bT} \right) - gT$$

This is not clearly positive or negative. It depends on the values of the various parameters.

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3. The tipping flagpole This problem has been excised due to being unphysical. It neglected sheering force, important for flagpoles but not so much for ropes!

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4. A MIRVed projectile A projectile of mass $2m$ is fired with an initial velocity v_0 at an angle of θ_0 to the horizontal. It has a timer so that at some point along its trajectory, it will explode horizontally into two bits, each of mass m. What should the timer be set to so that the total range of the forward-going projectile is the biggest? Challenge: Does anything change if the projectile explodes not horizontally but along the tangent vector to its path?

A little bit of thought before throwing math at this tells us that we want the first projectile to explode immediately. The idea is that because the vertical component doesn't change when the projectile explodes, the total time of flight of the projectile is invariant. So we want the horizontal speed to be as large as possible for the entire time of flight, so we want the projectile to explode immediately.

For the challenge part, yes the problem definitely does change and as far as I can tell becomes rather nontrivial. If there is a clever way to approach this, I haven't discovered it. Let me know if you come up with one!

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5. Goliath's doom A rope has linear mass density λ and length ℓ with a ball of mass m at the end. The rope is swung at an angular velocity ω. Find the tension at an arbitrary point r along the length of the rope. Neglect gravity.

Consider a small element dr of the rope, letting r denote the distance from this element to the axis of rotation. On the side closer to the axis, the tension is $T(r)$. On the other side, the tension is $T(r+dr)$. Accounting for the centrifugal force in the rotating frame – or having the difference of T's be the centripetal force – gives

$$T(r) - T(r+dr) = -T'(r)dr = \lambda dr\, \omega^2 r$$

Cancelling out the dr, we can easily integrate to get $T(r)$, noting that $T(\ell) = m\omega^2\ell$. A similar way to approach is to note

$$T(r) = -\frac{1}{2}\lambda\omega^2 r^2 + \text{constant}$$

Using $T(\ell)$'s value, we can determine the constant to give

$$T(r) = \frac{1}{2}\lambda\omega^2(\ell^2 - r^2) + m\omega^2\ell$$

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6. The sliding dumbbell Here is a modification of a classic problem in mechanics. A dumbbell with two point masses m and length ℓ is laid vertically against a wall. The bottom mass of nudged so that it begins to slide. We want to find the angle where the top mass comes away from the wall. Do this problem in the following way.

a) First suppose that the masses are constrained to move along the x and y axes. Let the angle θ be the angle between the dumbbell and the wall. (It is initially zero.) Write the FBD for the two masses and find the equation of motion for θ. You should find $\ddot{\theta} = \frac{g}{\ell}\sin\theta$ after a fair amount of work.

b) Using conservation of energy, show that $\omega^2 = \frac{2g}{\ell}(1-\cos\theta)$ where $\omega = d\theta/dt$.

c) In the real problem, the top mass comes away from the wall when the horizontal constraint force becomes negative. (Since a normal force can only push, not pull.) Find the angle when this occurs. (You should find $\cos\theta = 2/3$.)

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a. Denoting x to be the horizontal coordinate of the bottom mass and y be the vertical corodinate of the top mass, we note that $x = \ell\sin\theta$ and $y = \ell\cos\theta$ at an arbitrary angle θ after the masses have begun to move. The free body diagram for the bottom mass involves an N_2 pointing upward, a force mg pointing downward, and a "tension" T pointing down and to the right at an angle θ to the vertical. Using $F = ma$ for the two components of the bottom mass gives

$$m\ddot{x} = T\sin\theta, \quad 0 = N_2 - mg - T\cos\theta$$

For the top mass, we have N_1 pointing rightward, mg pointing downward, and a T pointing up and to the left. The components of the $F = ma$ equation gives

$$0 = N_1 - T\sin\theta, \quad m\ddot{y} = T\cos\theta - mg$$

Note that the equations imply that $N_1 = m\ddot{x}$, which is totally sensible. N_1 is the only horizontal external force applied to the system and so it must give the only horizontal acceleration, $m\ddot{x}$, that the system has.

Noting that

$$\ddot{x} = \ell\ddot{\theta}\cos\theta - \ell\dot{\theta}^2\sin\theta$$

and

$$\ddot{y} = -\ell\ddot{\theta}\sin\theta - \ell\dot{\theta}^2\cos\theta$$

and making use of

$$\ddot{x} = \tan\theta\ (\ddot{y} + g)$$

(which can be proven from the equations involving x and y by eliminating T), one can show that

$$\ddot{\theta} = \frac{g}{\ell}\sin\theta$$

b. The initial energy of the system is $mg\ell$. After it has rotated to an angle θ, the energy is

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + mg\ell\cos\theta = \frac{1}{2}m\ell\dot{\theta}^2 + mg\ell\cos\theta$$

which must equal $mg\ell$. Ths gives

$$\omega^2 = \frac{2g}{\ell}(1 - \cos\theta)$$

c. Recall that $N_1 = m\ddot{x}$. So long as this is positive, the top mass is in contact with the wall. However, once N_1 drops to zero the mass begins to lose contact. (This is more obvious if we note that if it maintained contact, N_1 would have to go negative, and this is clearly unphysical.) Doing some math gives

$$N_1 = m\ell\left(-\omega^2\sin\theta + \ddot{\theta}\cos\theta\right)$$

Plugging in the equation of motion for $\ddot{\theta}$ and also using the energy conservation formula for ω^2 gives us

$$N_1 = mg\sin\theta\,(-2 + 3\cos\theta)$$

Therefore, contact is lost when $\theta = \arccos(2/3)$.
