A Brief Review of K&K Chapters 1-10

This is a brief overview of some of the important material in the first ten chapters of Kleppner and Kolenkow. It is not comprehensive, and there are very few derivations or detailed explanations, so it is no substitute for your notes from class or the actual text. Nonetheless, I hope you will find this outline useful in preparing for Thursday's midterm...

1. Vectors and Kinematics

Let's begin with some basics about vectors that you surely know well by now. Consider two vectors $\vec{A}$ and $\vec{B}$, where

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k} \ , \tag{1.1}$$

$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k} \ . \tag{1.2}$$

The scalar or dot product of two vectors $\vec{A}$ and $\vec{B}$ is given by

$$\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z \tag{1.3}$$

$$= |\vec{A}||\vec{B}|\cos\theta \ , \tag{1.4}$$

where θ is the angle between the vectors. The vector or cross product is given by

$$\vec{A} \times \vec{B} = \vec{C} \ , \tag{1.5}$$

where $\vec{C}$ is orthogonal to both $\vec{A}$ and $\vec{B}$, pointing in a direction determined by the right-hand rule. The magnitude of $\vec{C}$ is given by

$$|\vec{C}| = |\vec{A}||\vec{B}|\sin\theta \ . \tag{1.6}$$

The vector $\vec{C}$ can also be found explicitly by taking the following determinant:

$$\vec{C} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \ . \tag{1.7}$$

Another important vector operation with which you need to be familiar is differentiation. The derivative of a vector $\vec{A}$ is given by

$$\frac{d\vec{A}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{A}(t + \Delta t) - \vec{A}(t)}{\Delta t} \ . \tag{1.8}$$

Note that $\frac{d\vec{A}}{dt}$ is a <u>new</u> vector, which means that $\vec{A}$ can change both in magnitude and direction.

Here are some useful identities for vector differentiation:

$$\frac{d}{dt}\left(c\vec{A}\right) = \frac{dc}{dt}\vec{A} + c\frac{d\vec{A}}{dt} \ , \tag{1.9}$$

$$\frac{d}{dt}\left(\vec{A}\cdot\vec{B}\right) = \frac{d\vec{A}}{dt}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{dt} \ , \tag{1.10}$$

$$\frac{d}{dt}\left(\vec{A}\times\vec{B}\right) = \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt} \ , \tag{1.11}$$

$$\frac{dA^2}{dt} = 2\vec{A}\cdot\frac{d\vec{A}}{dt} \ . \tag{1.12}$$

A very important specific case that has appeared in many, many problems is the representation of vectors using polar coordinates. The polar coordinate system actually moves with the "particle" (or whatever we're considering), so, unlike the cartesian basis vectors $(\hat{\imath}, \hat{\jmath}, \hat{k})$, the polar basis vectors $(\hat{r}, \hat{\theta})$ change in time:

$$\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta} \ , \tag{1.13}$$

$$\frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r} \ . \tag{1.14}$$

Using Eqs. (1.9), (1.13), and (1.14), we find for the position, velocity, and acceleration vectors:

$$\vec{r} = r\hat{r} \ , \tag{1.15}$$

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \ , \tag{1.16}$$

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta} \ . \tag{1.17}$$

These are really useful formulae for analyzing any "curved" motion (e.g., circular, elliptical, etc.).

Another essential mathematical tool that shows up in this chapter is the Taylor series. The idea here is that any arbitrary function $f(x)$ which is continuous and differentiable can be represented by a power series in x:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \ . \tag{1.18}$$

The coefficients a_n can be found in terms of the derivatives of $f(x)$ at some point x_0, and this gives the behavior of the function in the neighborhood of x_0:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_0)^2}{2} + f'''(x_0)\frac{(x - x_0)^3}{3} + \ldots \tag{1.19}$$

One of the more common applications of the Taylor series is the binomial series:

$$(1 + x)^n = 1 + nx + \frac{1}{2}n(n - 1)x^2 + \ldots \tag{1.20}$$

2. Newton's Laws

Newton's laws are:

(1) Inertial systems exist. There are certain reference frames in which bodies at rest tend to stay at rest and bodies in motion tend to stay in motion.

(2) The acceleration of a body (in an inertial frame) is given by the applied force divided by the object's mass, or, more commonly

$$\vec{F} = m\vec{a} \ . \tag{2.1}$$

(3) If body 1 exerts a force $\vec{F}$ on body 2, then there must be a force $-\vec{F}$ acting on body 2 due to body 1. Therefore, in an isolated system, all forces must sum to zero.

There's not too much to say about this chapter... it all becomes clear when we apply these principles to specific problems as we have done all semester in the homework... I recommend looking over pages 68-70 in K&K which give some pretty good advice on solving problems related to Newton's Laws.

Some common forces you should at least be familiar with are:

Gravitational force
The gravitational force $\vec{F}_g$ on a mass m due to a mass M is

$$\vec{F}_g = -\frac{GMm}{r^2}\hat{r} \ , \tag{2.2}$$

where $\hat{r}$ points from m to M, and $G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$. The gravitational force due to the earth is

$$\vec{F}_g = -mg\hat{r} \ , \tag{2.3}$$

since on the surface of the earth, $r = R_E$,

$$g = \frac{GM_E}{R_E^2} \ . \tag{2.4}$$

Electrostatic force
The electrostatic force $\vec{F}_e$ on charge q due to charge Q is

$$\vec{F}_e = k\frac{Qq}{r^2}\hat{r} \ , \tag{2.5}$$

where $k = 8.99 \times 10^9 \ rmN \cdot m^2/C^2$.

Tension in a string
We can think about a string as a bunch of short pieces, each pulling on the other (like a bunch of links in a chain). The tension is the magnitude of the force acting between adjacent pieces. See page 88 of K&K.

Normal force
Solid surfaces prevent bodies from “sinking” into them, so there is a force perpendicular to the surface which acts on any body. This is the normal force, and it exactly cancels any forces which would cause motion “into” the surface.

Friction force
Friction f opposes the motion of a body. For bodies not in relative motion

$$0 \leq f \leq \mu N \ , \tag{2.6}$$

where f exactly balances any forces that would cause motion. For bodies which are in relative motion

$$f = \mu' N \ . \tag{2.7}$$

The quantity μ is the coefficient of static friction and μ' is the coefficient of kinetic friction (μ' is a little less than μ).

Spring force
The restoring force from a spring is

$$F = -k\Delta x \ , \tag{2.8}$$

where k is the spring constant and Δx is the displacement from equilibrium. The frequency of oscillation ω due to the spring force is (see K&K page 99)

$$\omega = \sqrt{\frac{k}{m}} \ . \tag{2.9}$$

3. Momentum

The momentum $\vec{p}$ of an object is defined to be

$$\vec{p} = m\vec{v} \ . \tag{3.1}$$

Right away we note that

$$\vec{F} = \frac{d\vec{p}}{dt} \; . \tag{3.2}$$

This means that if there are no external forces acting on a system, then

$$\frac{d\vec{p}}{dt} = 0 \; , \tag{3.3}$$

so $\vec{p}$ is constant in time. This is the law of conservation of momentum. Because of Newton's third law, all of the internal forces cancel, and we can think about the motion of the center of mass $\vec{R}_{cm}$:

$$\vec{R}_{cm} = \frac{1}{M_{tot}} \sum_i m_i \vec{r_i} \; . \tag{3.4}$$

Then we have

$$\vec{F}_{ext} = M_{tot} \ddot{\vec{R}}_{cm} \; . \tag{3.5}$$

Of course, the internal forces can change the relative positions of the constituent parts of a body, leading to changes in the body's orientation in space. This is the basis for decomposing motion into translation of the center of mass and rotation about the center of mass. In integral form, Eq. (3.4) becomes

$$\vec{R}_{cm} = \frac{1}{M_{tot}} \int \vec{r} dm \; . \tag{3.6}$$

Equation (3.2) can be reformulated in terms of the impulse $\Delta\vec{p}$:

$$\int_0^t \vec{F} dt = \vec{p}(t) - \vec{p}(0) = \Delta\vec{p} \; , \tag{3.7}$$

which is good for considering "average forces" or net changes in momentum (like the ball bouncing between walls, K&K problem 4.29 from problem set 5, see also Section 3.6 in K&K).

Be sure to understand the "rocket problem" (K&K example 3.11, pages 134-5).

Also note that pressure P is defined to be the force divided by the area over which it acts:

$$P = \frac{F}{A} \; . \tag{3.8}$$

4. Work and Energy

The central result of chapter 4 is the Work-Energy theorem:

$$\int_{\vec{r_1}}^{\vec{r_2}} \vec{F} \cdot d\vec{r} = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 \; , \tag{4.1}$$

$$= K(\vec{r}_2) - K(\vec{r}_1) = \Delta K \; , \tag{4.2}$$

where K is the kinetic energy. The line integral

$$W = \int_{\vec{r_1}}^{\vec{r_2}} \vec{F} \cdot d\vec{r} \tag{4.3}$$

is known as the work.

For many forces, the work integral is independent of the path and depends only on the endpoints. Forces for which this is true are known as conservative forces. In this case, we can talk about the potential energy U:

$$\int_{\vec{r_1}}^{\vec{r_2}} \vec{F} \cdot d\vec{r} = -U(\vec{r}_2) + U(\vec{r}_1) \ , \tag{4.4}$$

so the work-energy theorem can be re-written

$$K(\vec{r}_1) + U(\vec{r}_1) = K(\vec{r}_2) + U(\vec{r}_2) \ , \tag{4.5}$$

which, in essence, is the law of conservation of energy, if we define the quantity $E = K + U$.

A conservative force can be found from the potential associated with it according to

$$\vec{F} = -\vec{\nabla} U \ , \tag{4.6}$$

where $\vec{\nabla}$ is the "del" operator discussed in K&K chapter 5.

Another question that commonly arises in the analysis of some physical system concerns the frequency of small oscillations about some equilibrium position r_0. The effective spring constant k can be found from the potential energy function according to

$$k = \left. \frac{d^2U}{dr^2} \right|_{r=r_0} \ . \tag{4.7}$$

Using this in Eq. (2.9) gives the frequency of small oscillations.

Another interesting and important point is that constraint forces (forces like those of railroad tracks which keep the train rolling in a particular direction) do no work, because the motion is always orthogonal to the direction of the force. To see this we can write:

$$d\vec{r} = \vec{v} dt \ , \tag{4.8}$$

and for constraint forces $\vec{F} \cdot \vec{v} = 0$.

If you are dealing with a nonconservative force like friction f, you can calculate the work it does and subtract it from the total energy:

$$\Delta E = \int \vec{f} \cdot d\vec{r} \ . \tag{4.9}$$

A useful concept in many applications is the idea of power P:

$$P = \frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{v} \ . \tag{4.10}$$

5. Mathematical Aspects of Force and Energy

A pretty useful mathematical operator is $\vec{\nabla}$:

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \ , \tag{5.1}$$

which when operated on a scalar function gives the gradient. You can also form the cross product of $\vec{\nabla}$ with a vector function, which gives the curl:

$$\vec{\nabla} \times \vec{F} = \lim_{a \to 0} \frac{\oint_\Gamma \vec{F} \cdot d\vec{\ell}}{a} \ , \tag{5.2}$$

where a is the area enclosed by the closed path Γ. This is the infinitesimal "circulation" of the function $\vec{F}$. The relation (5.2) leads directly to Stokes' Theorem:

$$\oint \vec{F} \cdot d\vec{r} = \int \left(\vec{\nabla} \times \vec{F}\right) \cdot d\vec{a} \ . \tag{5.3}$$

If

$$\vec{\nabla} \times \vec{F} = 0 \ , \tag{5.4}$$

then a force $\vec{F}$ is conservative, because according to Stokes' theorem the work done is path-independent.

6. Angular Momentum and Fixed Axis Rotation

This is probably the most important chapter covered since the last midterm. Most of the material in chapters 7-9 depend upon understanding chapter 6, so make sure you're very comfortable with this stuff.

Well, the place to start is with the definition of angular momentum $\vec{L}$:

$$\vec{L} = \vec{r} \times \vec{p} \ , \tag{6.1}$$

where $\vec{r}$ is the distance of the object from the origin of a particular coordinate system and $\vec{p}$ is the momentum with respect to that coordinate system. A very crucial point is that $\vec{L}$ depends on the coordinate system! Also note that $\vec{L}$ points in a direction orthogonal to $\vec{r}$ and $\vec{p}$. These two qualities of angular momentum often make for confusing problems.

The angular momentum of an object is changed through torques $\vec{\tau}$:

$$\vec{\tau} = \vec{r} \times \vec{F} \ , \tag{6.2}$$

and

$$\vec{\tau} = \frac{d\vec{L}}{dt} \ . \tag{6.3}$$

So we see that if there are no external torques on an object, then the angular momentum doesn't change, i.e.,

$$\frac{d\vec{L}}{dt} = 0 \ , \tag{6.4}$$

which gives us the law of conservation of angular momentum.

Many of the problems we have seen deal with the motion of rigid bodies. K&K start by considering fixed axis rotation, where the direction of the axis of rotation is held constant. The angular momentum of the i^{th} particle in some body is

$$L_i = m_i v_i r_i \tag{6.5}$$

where r_i is the distance to the rotation axis. Since

$$v_i = \omega r_i \ , \tag{6.6}$$

we have

$$L_i = m_i r_i^2 \omega \ . \tag{6.7}$$

Thus it is natural to define the moment of inertia I for the rigid body:

$$I = \sum_i m_i r_i^2 \ , \tag{6.8}$$

or in integral form

$$I = \int r^2 dm \ . \tag{6.9}$$

Thus, for the angular momentum of the entire body

$$L = I\omega \ . \tag{6.10}$$

The moments of inertia for some simple bodies are:

- Uniform thin hoop of mass M and radius R, axis through center and perpendicular to plane of hoop: $I = MR^2$.
- Uniform thin hoop of mass M and radius R, axis along diameter (in the plane of the hoop): $I = \frac{1}{2}MR^2$.
- Uniform disk of mass M, radius R, axis through center and perpendicular to plane of disk: $I = \frac{1}{2}MR^2$.
- Uniform thin stick of mass M, length ℓ, axis through midpoint and perpendicular to stick: $I = \frac{1}{12}M\ell^2$.
- Uniform thin stick of mass M, length ℓ, axis at one end and perpendicular to stick: $I = \frac{1}{3}M\ell^2$.
- Uniform sphere of mass M and radius R, axis through center: $I = \frac{2}{5}MR^2$.

A handy result is the parallel axis theorem, which states that if we know I_0, the moment of inertia about an axis through the center of mass, then the moment of inertia I about any parallel axis is given by

$$I = I_0 + MR^2 \ , \tag{6.11}$$

where R is the distance between the two axes.

An extremely useful technique is to decompose motion into translation of the center of mass and rotation about an axis through the center of mass. When we do this, we find that the kinetic energy of an object can be written:

$$K = \frac{1}{2}Mv^2 + \frac{1}{2}I_0\omega^2 \ . \tag{6.12}$$

Seems simple enough... but this stuff usually turns out to be quite tricky. I recommend looking over problem set 6.

7. Rigid Body Motion

If a rigid body is rotating about some axis with angular velocity $\vec{\omega}$, then the velocity of some part of the rigid body a distance $\vec{r}$ from the origin is given by

$$\frac{d\vec{r}}{dt} = \vec{v} = \vec{\omega} \times \vec{r} \ . \tag{7.1}$$

Much of this chapter deals with gyroscope motion. The essential equation for gyroscope motion states that the angular momentum of the gyroscope L_s always points along the axis, and external torques then cause the direction of the axis to precess, but don't change the magnitude of L_s:

$$\left| \frac{d\vec{L}_s}{dt} \right| = \Omega L_s \ , \tag{7.2}$$

where Ω is the precession frequency.

This chapter also raises another curious point. The moment of inertia of a rigid body is in fact a tensor, so we must write

$$\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega} \ , \tag{7.3}$$

where

$$\overleftrightarrow{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \ . \tag{7.4}$$

This is what leads to such funny phenomena as "torque-free precession."

8. Noninertial systems and Fictitious Forces

Newton's laws only hold in inertial frames. If your frame is accelerating, then from your perspective it looks like objects are accelerating with no apparent forces acting on them. We can use "fictitious forces" to account for this apparent motion, and then happily apply Newton's laws as if nothing is wrong.

A uniformly accelerating system, with acceleration $\vec{a}$, has the fictitious force

$$\vec{F}_{fic} = -m\vec{a} \ . \tag{8.1}$$

For a rotating system with angular velocity $\vec{\Omega}$, $\vec{a}$ is

$$\vec{a} = 2\vec{\Omega} \times \vec{v} + \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right) \ . \tag{8.2}$$

The first term leads to the coriolis force and the second term gives us the centrifugal force.

9. Central Force Motion

For a central force between two bodies $F(r)$, we can reduce it to a one-body problem by introducing the reduced mass μ

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \ , \tag{9.1}$$

then in terms if the distance between the two masses $\vec{r}$, we can write

$$\mu\ddot{\vec{r}} = F(r)\hat{r} \ . \tag{9.2}$$

Because the central force $F(r)$ exerts no torque on the bodies (measured from the center of mass), the angular momentum L is constant. The central force is also conservative, so the energy E is constant. These two ideas allow us to write

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{L^2}{\mu r^2} + U(r) \ , \tag{9.3}$$

$$= \frac{1}{2}\mu\dot{r}^2 + U_{\text{eff}}(r) \ . \tag{9.4}$$

We can use this equation to get r as a function of t:

$$\frac{dr}{dt} = \sqrt{\frac{2}{\mu}(E - U_{\text{eff}}(r))} \ , \tag{9.5}$$

which can be integrated in some cases. One can also do some manipulations to get

$$\frac{d\theta}{dr} = \frac{L}{\mu r^2}\frac{1}{\sqrt{\frac{2}{\mu}(E - U_{\text{eff}}(r))}} \ . \tag{9.6}$$

Make sure you understand the energy diagrams on page 386 of K&K.

A big topic in this chapter is planetary motion (I guess it's also cometary motion, but, whatever). We have some sort of central force for which the potential is

$$U(r) = -\frac{C}{r} \ , \tag{9.7}$$

for which the solution of r as a function of θ is

$$r = \frac{r_0}{1 - \epsilon \cos\theta} \ , \tag{9.8}$$

where

$$r_0 = \frac{L^2}{\mu C} \tag{9.9}$$

and the eccentricity ϵ is

$$\epsilon = \sqrt{1 + \frac{2EL^2}{\mu C^2}} \ . \tag{9.10}$$

If $\epsilon > 1$, the motion is hyperbolic. If $\epsilon = 1$, it is a circular orbit. If $\epsilon < 1$ we have an elliptical orbit.

For elliptical orbits, the minimum value of r is the perigee r_p, for which

$$r_p = \frac{r_0}{1 + \epsilon} \ , \tag{9.11}$$

the maximum value of r is the apogee of the orbit

$$r_a = \frac{r_0}{1 - \epsilon} \ . \tag{9.12}$$

The length of the major axis of the ellipse A is given by

$$A = r_p + r_a = \frac{2r_0}{1 - \epsilon^2} \ . \tag{9.13}$$

A really cool result is Kepler's third law, which relates the period of the orbit T to the major axis of the ellipse

$$T^2 = kA^3 \ , \tag{9.14}$$

where k is a constant.

10. The Harmonic Oscillator

If you love physics, you've got to love the harmonic oscillator. That's because everywhere you look, there it is. Why? Well, many physical systems that we study are in equilibrium, and if they're not in equilibrium, they're on their way. We saw way back in Eq. (4.7) that the motion of a system about a stable equilibrium can be put into analogy with the simple harmonic motion of a mass on a spring.

Let's not beat around the bush, and get right to the general case of a free, damped oscillator. If the frictional force f is

$$f = -bv \;, \tag{10.1}$$

then we get the differential equation

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \;, \tag{10.2}$$

where

$$\gamma = \frac{b}{m} \;, \tag{10.3}$$

$$\omega_0 = \sqrt{\frac{k}{m}} \;. \tag{10.4}$$

I like to solve this equation with the complex guess

$$x(t) = Ae^{\alpha t} \;, \tag{10.5}$$

where α is complex and we'll take the real part later. Plugging our guess (10.5) into the differential equation (10.2), we get

$$\alpha^2 + \alpha\gamma + \omega_0^2 = 0 \;. \tag{10.6}$$

Therefore, from the quadratic equation,

$$\alpha = -\frac{\gamma}{2} \pm i\sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \;. \tag{10.7}$$

Therefore x oscillates at the frequency

$$\omega_1 = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \;, \tag{10.8}$$

and decreases in magnitude according to $e^{-(\gamma/2)t}$.

The quality factor Q of an oscillator is defined to be the energy stored in the oscillator divided by the energy dissipated per radian, and is given by

$$Q = \frac{\omega_1}{\gamma} \approx \frac{\omega_0}{\gamma} \;. \tag{10.9}$$

We can also subject an oscillator to a driving force F_d. Let's suppose $F_d = F_0 e^{i\omega t}$. Then the differential equation for our system is

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i\omega t} \ . \tag{10.10}$$

Now we guess the solution

$$x(t) = Ae^{i(\omega t + \phi)} \ . \tag{10.11}$$

Plugging (10.11) into (10.10), we get

$$A = \frac{F_0}{m} \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} \tag{10.12}$$

$$\phi = \tan^{-1}\left(\frac{\gamma\omega}{\omega^2 - \omega_0^2}\right) \ . \tag{10.13}$$

From these equations, we observe the phenomenon of resonance. This is a really crucial concept. See pages 426-8 of K&K for a good example.