Two- and Three- level systems and All That

In order to understand oscillation, waves, and all kinds of other phenomena, it is essential to know your mass on a spring. A system of similar fundamental importance in atomic physics is the 2-level system (although many would argue that this is again nothing else but a mass on a spring).

In any event, we have our 2-levels (Fig. 1), all atoms are in the ground state A at t=0. We want to know what is the system's evolution if we apply a laser field of frequency ω_L.

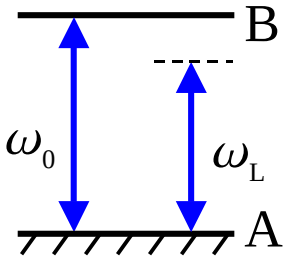

Fig. 1

The Hamiltonian of the light-atom interaction in the E1-approximation is:

$$\hat{V} = -\vec{d}\cdot\vec{E} \rightarrow dE_0 e^{i\omega_L t}. \qquad (0)$$

transition dipole moment — electric field of the light

We will use the matrix notation:

$$\psi = \begin{pmatrix} c_A \\ c_B \end{pmatrix} \text{ meaning } \psi = c_A\psi_A + c_B\psi_B = c_A|A\rangle + c_B|B\rangle. \qquad (1)$$

The field-free Hamiltonian is:

$$\hat{H}_0 = \begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} 0 & 0 \\ 0 & \omega_0 \end{pmatrix} \end{array} \quad (\hbar=1, \text{ naturally}), \qquad (2)$$

and the full Hamiltonian is:

$$\hat{H} = \hat{H}_0 + \hat{V} = \begin{pmatrix} 0 & dE \\ (dE)^* & \omega_0 \end{pmatrix}. \qquad (3)$$

Now the Schroedinger equation reads:

$$i\dot{\psi} = \hat{H}\psi \quad \Rightarrow \quad i\begin{pmatrix} \dot{c}_A \\ \dot{c}_B \end{pmatrix} = \begin{pmatrix} 0 & dE \\ (dE)^* & \omega_0 \end{pmatrix}\begin{pmatrix} c_A \\ c_B \end{pmatrix}. \qquad (4)$$

With the initial condition $\psi(0)=\begin{pmatrix}1\\0\end{pmatrix}$, this is enough for performing numerical calculations. You can play with this using the online Mathematica™ tutorial that is available on the course web site.

For this simple case, the analytical solution is also possible (see Ramsey, "Molecular Beams", p.119).

The most simple case is that of $\omega_L=\omega_0$. In this case,

$$|c_A|^2=\cos^2(dE_0t);\quad |c_B|^2=\sin^2(dE_0t). \tag{5}$$

This is the well-known Rabi oscillation. The formulae are still manageable even for non-zero detuning:

$$|c_B|^2=\frac{(2dE_0)^2}{(\omega_0-\omega_L)^2+(2dE_0)^2}\sin^2\left(\frac{\sqrt{(\omega_0-\omega_L)^2+(2dE_0)^2}}{2}t\right). \tag{6}$$

This shows that the "Rabi frequency" is

$$\Omega_R=\sqrt{d^2E_0^2+\frac{(\omega_0-\omega_L)^2}{4}}\quad , \tag{7}$$

i.e. the frequency of oscillation increases with detuning. On the other hand, the amplitude of the oscillation falls with detuning (see eq. (6)).

It is very important to understand how these results modify if one includes decay of the upper state. The most important result is that when the decay rate γ_0 becomes sufficiently large (with respect to dE_0), instead of an oscillatory solution, one gets an overdamped solution. This can be further explored in the online tutorial.

Adiabatic Passage

Perhaps it is a good time to mention that our "optical" 2-level system also describes a particle with angular momentum 1/2 and an associated magnetic moment in a magnetic field. Actually, many ideas from this realm (nuclear magnetic resonance, NMR) were very successfully applied in optics, including seminal contributions by a pioneer in both fields, Prof. Erwin Hahn of our Department.

Let us consider a classic NMR problem in which you have an ensemble of spin 1/2 particles, all of which are "spin down" in a dc magnetic field (i.e. in the state A in our

picture). Our task is to flip all the spins, i.e. put all of them into state B. How can this be done?

The first idea that may come to mind is to apply an oscillating auxiliary magnetic field with frequency in resonance with the A→B transition. It is clear that, in order to flip the spins, the direction of this field has to be orthogonal to that of our "leading" dc field. It is actually easier to consider the auxiliary field not as an oscillating field, but as a field rotating with the Larmor frequency $\omega_A=\Omega_L=g\mu B_0$, where g is the Landé factor, and B_0 is the leading field (Fig. 2). Going to the frame rotating with this frequency, we eliminate the leading field, so our spins just see the auxiliary field. Moreover, they see this field as dc. Now our spins precess around the auxiliary field with frequency

$$\Omega_R=g\mu B_A\,. \qquad (8)$$

It is not an accident that we write this as Ω_R. Indeed, this is a direct analog of the Rabi frequency we have just discussed.

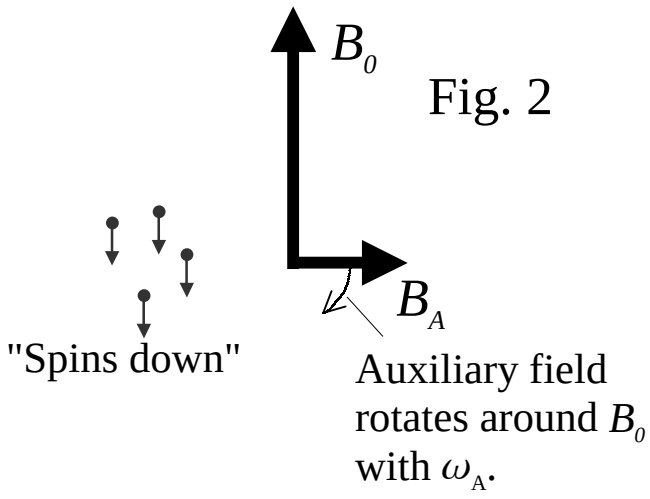

Fig. 2

Now, to get all spins that are originally down to flip, we can just apply our B_A for a time τ_π, so $\Omega_R\cdot\tau_\pi=\pi$, i.e. all spins are now pointing up. This is called a "π"-pulse.

This is an OK method, but it is ***not*** very robust. In fact, we have to fine-tune the product $B_A\cdot\tau$ to the π-pulse condition. This may also run into additional problems if not all particles in our ensemble see exactly the same magnetic fields.

Can we do better?

I would not be asking if we couldn't! Suppose that instead of using a field B_A rotating at the resonance frequency $\omega_A=\Omega_L$, we start with $\omega_A\ll\Omega_L$, and then sweep it smoothly to $\omega_A\gg\Omega_L$. (Actually, it would work equally well if we started with $\omega_A\gg\Omega_L$ and then swept it to $\omega_A\ll\Omega_L$, or if we change Ω_L instead by adjusting the leading field B_o).

Turns out, that under certain rather nonrestrictive conditions which we will specify in a moment, this operation leads to complete inversion of all spins, which is extremely robust, i.e. does not depend on the detials of the sweep: sweep speed, exact value of B_A, etc.

Let's see how this comes about. It is convenient to work in a frame co-rotating with B_A, in which our spins see an effective magnetic field:

$$\vec{B}_{eff} = \vec{B}_A + \left(\vec{B}_0 - \frac{\vec{\omega}_A}{g\mu} \right). \qquad (9)$$

Note that this equation shows explicitly that for $\omega_A=\Omega_L=g\mu B_0$, going to the rotating frame "eliminates" the leading field B_0 (of course, we have chosen the geometry so that $\vec{\omega}_A \parallel \vec{B}_0$).

OK, now we can sketch how the effective field looks in the rotating frame as a function of time as we sweep ω_A: (Fig. 3).

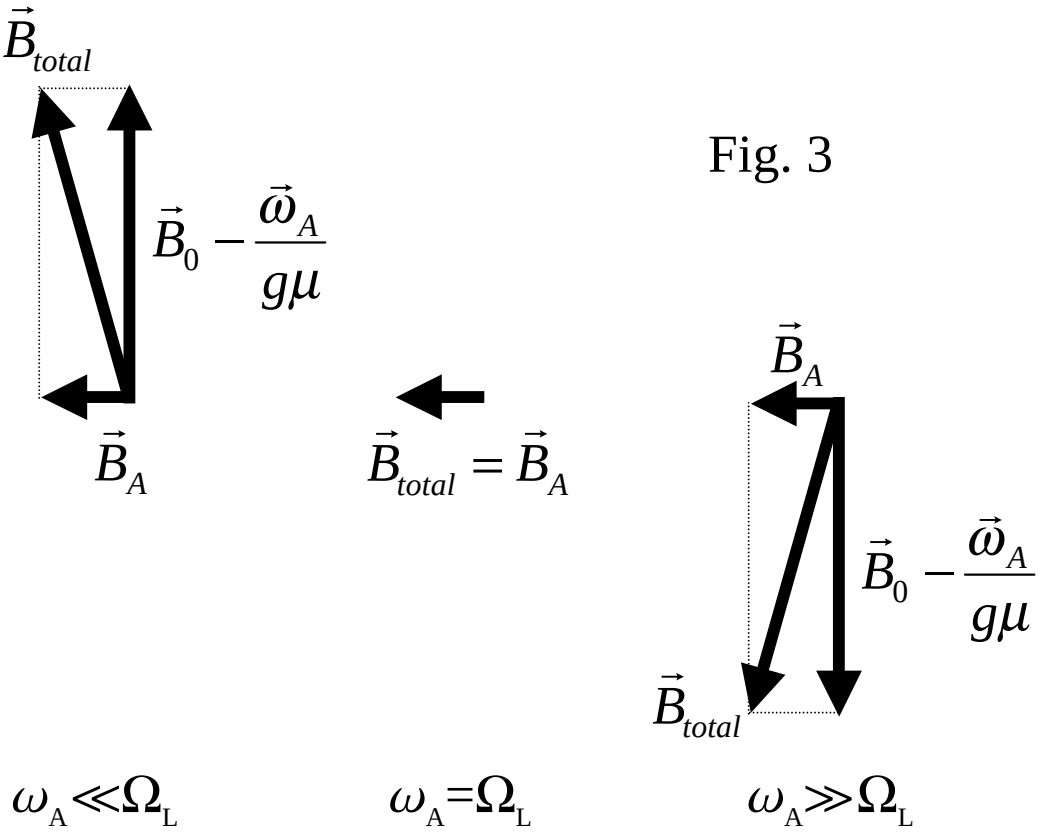

Fig. 3

How do spins behave in such a time dependent field? We know that if we change the field slowly enough, the spins follow adiabatically, i.e., if they were pointing essentially opposite to $\vec{B}_{total}$, they remain that way. Our rotating frame picture tells us then, that this procedure flips all spins.

What are the conditions on fields that ensure adiabaticity?

Qualitatively, the spins have to precess around $\vec{B}_{total}$ many times on the time scale $\left|\vec{B}_{total}\right| \Big/ \left|\dot{\vec{B}}_{total}\right|$, i.e. a time interval during which there is significant change in the field:

$$g\mu B_{total}\cdot\frac{\left|\vec{B}_{total}\right|}{\left|\dot{\vec{B}}_{total}\right|}\gg 1. \qquad (10)$$

If the sweep rate is constant, the most "dangerous" place is when we go though the resonance and $\vec{B}_{total}\approx\vec{B}_A$.

Therefore, with (8) we have:

$$\boxed{\Omega_R\cdot\frac{\Omega_R}{\dot{\omega}_A}>>1}. \qquad \text{Adiabatic Condition} \qquad (11)$$

Now let's see how this method can be applied in the optical domain.

Say we have an atomic beam of two-level atoms, which are all in the ground state.

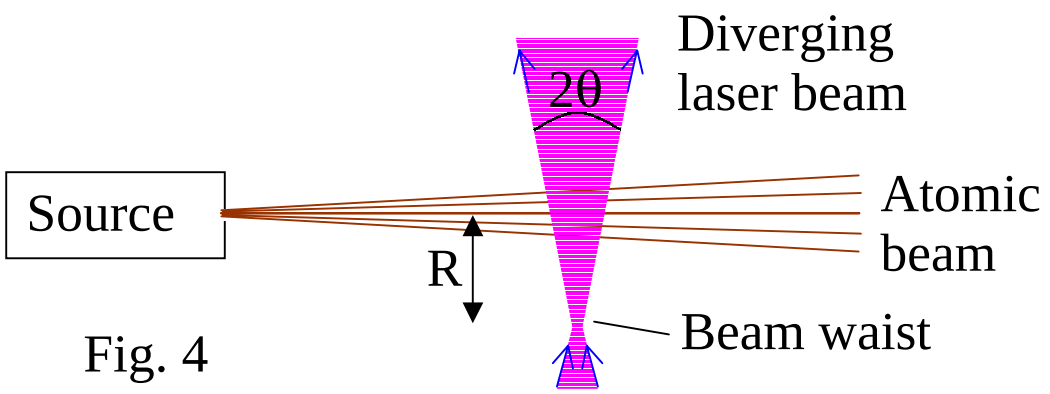

Fig. 4

We need to invert the atoms, i.e. put them into the upper state (notice the one-to-one correspondence with the spin-1/2 in a magnetic field). An elegant method to achieve this is to shine a diverging laser beam with frequency tuned to resonance onto the atoms as shown in Fig. 4. As an atom flies into the laser beam, it first sees light that is up-shifted from resonance by $\sim\frac{v\cdot\theta}{c}\cdot\omega_0$, where v is a characteristic speed of atoms in the beam. As an atom flies through the beam, the frequency goes through resonance, and becomes $\approx\omega_0-\frac{v\cdot\theta}{c}\cdot\omega_0$ as the atom leaves. The adiabatic condition (11) can be now written if we recall that in this case (see (7))

$$\Omega_R=dE\,, \qquad (12)$$

and we also use the fact that the quantity analogous to $\dot{\omega}_A$ in eqn. (11) is now

$$\dot{\omega} \approx \frac{\Delta\omega}{\Delta t} \approx \frac{\frac{v\theta}{c}\cdot\omega_0}{\frac{R\theta}{v}} = \frac{v^2\omega_0}{cR} \quad . \tag{13}$$

Note that we also need to make sure that the light-atom interaction time is shorter than the upper state lifetime. Adiabatic passage inversion using this method is used in our group in experiments with dysprosium (A.T. Nguyen et al).

Three-level Systems

Suppose now that we have a three-level system (Fig.5) and two pulsed lasers. One is tuned on resonance with the A→B transition, and the other – in resonance with B→C. Both laser pulses are sufficiently strong to saturate their respective transitions, and their bandwidths are greater than the transition widths. The questions are:

- What is the optimum pulse sequence?
- What is the maximum amount of atoms that we can put into C?
- How will this amount change if state B is "lossy" i.e. has large decay rate?

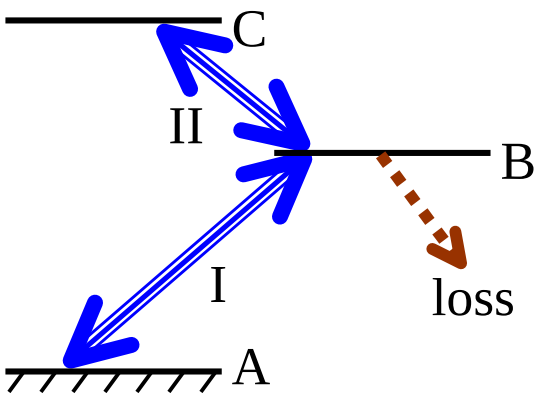

Fig. 5

The answers may seem surprising at first glance. It turns out that it is possible to put all atoms into C, even when B is lossy. To achieve this, one has to use the "counter-intuitive" pulse sequence (Fig.6), i.e. pulse II driving the B→C transition has to come first. How this comes about is well explained in the articles in the Reader. The basic idea is this. Consider first the time when pulse II is already on, but pulse I is not yet on. Out of 3 atomic states A, B, C, we have two interacting with light (B, C), and one "dark" (A). Our atoms happen to reside in the dark state. Now pulse I gradually turns on and we have a light field consisting of two frequencies. However, it turns out, there is still a dark coherent combination of A, B, and C. Moreover, atoms adiabatically follow the field change remaining in the dark state. As pulse II subsides, and we only have pulse I left, the dark state evolves into C,

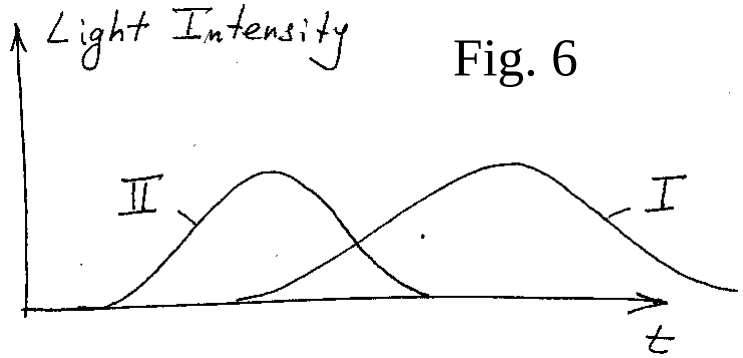

Fig. 6

and at the end of the day, or at least at the end of the pulse sequence, we find our atoms in C. Upon this discussion, we can rename this pulse sequence (Fig.6) as “intuitive”.