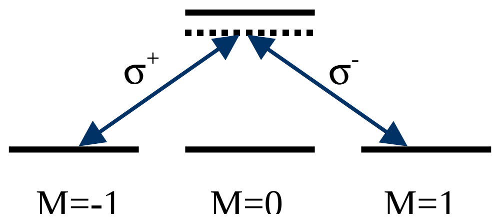
In this Note, we use a J=1 $\rightarrow$ J=0 transition to illustrate the origin of the *dark states* and also to show how one does a "rigorous" calculation of atomic transition amplitudes. A good reference for formulae used in this kind of calculations is: I. I. Sobleman, *Atomic Spectra and Radiative Transitions*, Springer-Verlag, New York, 1992.

The energy levels are shown in the figure. The light-atom interaction Hamiltonian for an electric dipole transition is ($\hbar$=1):




$$\hat{H} = -\vec{d} \cdot \vec{E} = -\left(d_x E_x + d_y E_y + d_z E_z\right), \tag{1}$$

where $\vec{d}$ is the transition dipole moment, and $\vec{E}$ is the light electric field.

It turns out to be convenient to switch from the Cartesian basis to a *spherical basis*. To do this, for each vector $\vec{v}$, we construct the following combination of the Cartesian components:

$$v_0 = v_z, \quad v_\pm = \frac{\mp(v_x \pm i v_y)}{\sqrt{2}}. \tag{2}$$

Correspondingly, we have:

$$v_x = \frac{(v_- - v_+)}{\sqrt{2}}, \quad v_y = i \cdot \frac{(v_+ + v_-)}{\sqrt{2}}. \tag{3}$$

Next, we explicitly apply Eqns. (2, 3) to the electric dipole moment and the electric field in Eqn. (1). This yields for the Hamiltonian:

$$\hat{H} = -\left(d_0 E_0 - d_+ E_- - d_- E_+\right). \tag{4}$$

Note that we have just derived a rule for taking scalar products of vectors represented in a spherical basis:

$$\vec{v} \cdot \vec{u} = \sum_{q=-1}^{1} (-1)^q v_q u_{-q} \ . \tag{5}$$

Returning to our problem, let us assume for concreteness that we have linearly polarized light with polarization along **x**:

$$\vec{E} = E_0 \cdot \hat{x} \cdot \cos(\omega t) = E_0 \cdot \cos(\omega t) \cdot \frac{\hat{e}_+ - \hat{e}_-}{\sqrt{2}}; \quad E_\pm = \pm \frac{E_0 \cdot \cos(\omega t)}{\sqrt{2}} \cdot \hat{e}_\pm \ . \tag{6}$$

From Eqns. (4,6) we have:

$$\hat{H} = d_+ E_- + d_- E_+ = \frac{E_0 \cdot \cos(\omega t)}{\sqrt{2}} \left( -d_+ \hat{e}_- + d_- \hat{e}_+ \right) . \tag{7}$$

The transition amplitude is given by:

$$A = \langle \psi_f | \hat{H} | \psi_i \rangle \propto C_{-1} \langle J = 0 | d_+ | J = 1, M = -1 \rangle + C_1 \langle J = 0 | d_- | J = 1, M = 1 \rangle . \tag{8}$$

Here, we used the well-known selection rules for the dipole transitions, and only wrote the non-zero matrix elements. Also (**very important!**), we assumed that the initial J=1 state is described as a *coherent superposition* of the sublevels:

$$\psi_i = C_{-1} | J = 1, M = -1 \rangle + C_1 | J = 1, M = 1 \rangle . \tag{9}$$

The two matrix elements in Eqn. (8) are related between each other according to the *Wigner-Eckart Theorem* which relates matrix elements between given Zeeman sublevels to a *reduced matrix element*. The general statement of the theorem is:

$$\langle \gamma JM | T_{\kappa q} | \gamma' J' M' \rangle = (-1)^{J-M} (\gamma J \parallel T_{\kappa} \parallel \gamma' J') \begin{pmatrix} J & \kappa & J' \\ -M & q & M' \end{pmatrix}. \tag{10}$$

Here $\gamma$ and $\gamma'$ represent all "other" quantum numbers, $(\gamma J \parallel T_{\kappa} \parallel \gamma' J')$ is the reduced matrix element (independent of *M*, *M*'); $\kappa$ and *q* are the tesor indices of the operator *T* ($\kappa$ being the *rank*), and the matrix in parentheses stands for a *Wigner 3J-symbol* which is related in a straightforward way to a Clebsch-Gordan coefficient:

$$(j_1 m_1 j_2 m_2 \mid jm) = (-1)^{-j1+j2-m} \sqrt{2j+1} \cdot \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix}. \tag{11}$$

3J-symbols, as well as Clebsch-Gordan coefficients are tabulated, and also built in programs like Mathematica™. A note of caution: there exist in the literature at least two different definitions of the reduced matrix elements, different by a factor like $\sqrt{2j+1}$. To avoid errors, I recommend consistently using the definitions used in Sobel'man's book.

For the dipole operator, we have $\kappa$=1, and the 3J-symbols relevant for Eqn. (8) are:

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{3}}. \tag{12}$$

Thus, we see that the transition amplitude *A* in the case of **x**-polarized light is proportional to the quantity $C_{-1} + C_1$. A state for which this quantity is zero is a *dark state* which does not interact with the light in any way. Repeating these calculations for different polarization, one can verify that for a 1→0 transition, we have:

$|J = 1, M = 0\rangle$ - **z**-absorbing, dark for **x**- and **y**- polarized light

$\frac{|J=1,M=1\rangle+|J=1,M=-1\rangle}{\sqrt{2}}$ - **y**-absorbing, dark for **z**- and **x**- polarized light

$\frac{|J=1,M=1\rangle-|J=1,M=-1\rangle}{\sqrt{2}}$ - **x**-absorbing, dark for **z**- and **y**- polarized light

In these expressions, the factor in the denominator is included for normalization.

Dark states play an important role in nonlinear optical rotation, electromagnetically induced transparency, subrecoil laser cooling, and many other applications of modern atomic physics.