"The purpose of physics is to understand the universe... the purpose of mathematics is, well, obscure to me..."

- Prof. Seamus Davis, UC Berkeley

I would like to emphasize that the somewhat controversial opinions expressed in the solution to problem 10 are entirely the fault of the present author, but were arrived at with great assistance from Prof. Budker, of course.

If you have any questions, suggestions or corrections to the solutions, don't hesitate to e-mail me at dfk@uclink4.berkeley.edu!

Problem 8 *Bose-Einstein condensation temperature*

The DeBroglie wavelength of a particle is given by:

$$\lambda_{DB} = \frac{2\pi\hbar}{Mv}, \tag{1}$$

where M is the mass of the particle and v is its velocity. The thermal velocity of a particle can be found from the equipartition theorem:

$$\frac{1}{2}Mv^2 = \frac{3}{2}k_BT, \tag{2}$$

where T is the temperature and k_B is Boltzmann's constant. Thus the DeBroglie wavelength as a function of temperature is given by:

$$\lambda_{DB} = \frac{2\pi\hbar}{\sqrt{3k_BMT}}. \tag{3}$$

The DeBroglie wavelength is equal to the interparticle separation when:

$$\lambda_{DB} = n^{-1/3}, \tag{4}$$

where n is the particle density, $n = 10^9$ cm^{-3} in this problem. From Eqns. (3) and (4) we deduce the condition for the Bose-Einstein condensation temperature T_c:

$$\boxed{T_c = \frac{(2\pi\hbar)^2 n^{2/3}}{3k_BM}}$$

Numerically, we find:

$$T_c = \frac{(2\pi)^2 \cdot (6.6\times10^{-16}\ \text{eV}\cdot\text{s})^2 \cdot (10^6\ \text{cm}^{-2}) \cdot (3\times10^{10}\ \text{cm/s})^2}{3\cdot(8.6\times10^{-5}\ \text{eV/K})\cdot(23\cdot931\times10^6\ \text{eV})} \approx 3\times10^{-9}\ \text{K}.$$

An exact solution gives

$$\boxed{T_c = \frac{3.31\hbar^2 n^{2/3}}{k_BM} \approx 1\times10^{-9}\ \text{K}}$$

Problem 9 *Laser Cooling vs. Light-Induced Drift*

(a)

The typical force imparted to an atom in laser cooling is the photon momentum $\hbar k$ divided by the lifetime of the upper state τ:

$$F_{\text{LC}} = \frac{\Delta p}{\Delta t} \sim \frac{\hbar k}{\tau} = \frac{\hbar\omega}{c\tau} \sim \frac{1.6\times10^{-12}\ \text{erg}}{(3\times10^{10}\ \text{cm/s})\cdot(10^{-8}\ \text{s})} \approx 5\times10^{-15}\ \text{dyne},$$

where we choose $\hbar\omega \approx 1$ eV $= 1.6\times10^{-12}$ erg as the typical atomic energy scale and $\tau \approx 10^{-8}$ s as the typical atomic lifetime.

(b)

The typical force imparted to an atom in light-induced drift (see pp. 131-136 in the Reader) is given by the difference between the frictional forces for ground state and excited state atoms. For an ensemble of atoms with a given velocity (which is selected via the tuning of the laser light), collisions with an atomic gas in equilibrium tend to reduce the average momentum of the atomic ensemble to zero. Therefore $\Delta p \sim M\bar{v}_{\text{res}}$, where $\bar{v}_{\text{res}}$ is the average velocity of atoms resonant with the light. Here we assume a relatively heavy buffer gas. The frequency of such collisions (assuming atoms spend about half the time in the excited state) is

$$\gamma_{\text{col}} \sim \frac{1}{2}n\sigma^*\bar{v}_{\text{res}},$$

where σ^* is the collisional cross section for the excited state. Roughly, then, we estimate the force on resonant atoms to be:

$$F_{\text{LID}} = \frac{\Delta p}{\Delta t} \approx \gamma_{\text{col}}\Delta p \sim \frac{1}{2}nM\sigma^*\bar{v}_{\text{res}}^2.$$

We choose $n = 3\cdot10^{15}$ cm^{-3}, $\sigma^* \approx 10^{-13}$ cm^2, and $\bar{v}_{\text{res}} \approx 10^5$ cm/s as typical values. Then

$$F_{\text{LID}} \sim 10^{-10}\ \text{dyne}.$$

This is more than four orders of magnitude larger than the typical forces in laser cooling!

Problem 10 *Bose-Einstein Statistics for Photons*

The spin-statistics theorem, which states that integer spin particles must be in a symmetric state (with respect to particle interchange) and half-integer spin particles must be in an antisymmetric state, can be derived from relativistic quantum field theory using assumptions of local Lorentz invariance, causality and the existence of anti-particle fields [see, e.g., S. Weinberg, *The Quantum Theory of Fields*, (Cambridge University Press, Cambridge, 1995), pp. 229-238]. A violation of the spin-statistics theorem would imply a violation of one of these assumptions of relativistic quantum mechanics. Such a violation is not totally ruled out by experiment: for example, there are a number of ongoing experiments to test local Lorentz invariance [e.g., C.J. Berglund, L.R. Hunter, D. Krause Jr., E.O. Prigge, M.S. Ronfeldt, and S.K. Lamoreaux, Phys. Rev. Lett. ƒ75, 1879 (1995)]. It is difficult to say how strictly tests of the assumptions of relativistic quantum mechanics limit violations of the spin-statistics theorem without reference to a specific model. Experimental tests of the spin-statistics theorem can therefore probe the validity of some of the most fundamental assumptions in physics. It should also be remembered that relativistic quantum mechanics is not a final theory of all phenomena, and is therefore inherently incomplete (just as classical mechanics is superseded by special relativity, relativistic quantum mechanics will likely be superseded by some deeper theory). There have been a few plausible scenarios for small spin-statistics violations proposed, such as excitations of higher dimensions (as in string theories) allowing particles to possess wrong-symmetry states in the usual 3-dimensional space while maintaining the correct symmetry in the N-dimensional space [O.W. Greenberg and R.N. Mohapatra, Phys. Rev. D **39**, 2032 (1989)]. Theoretical attempts to incorporate a small violation of the spin-statistics theorem into standard quantum mechanics, most notably the quon algebra [O.W. Greenberg, Phys. Rev. Lett. **64**, 705 (1990)], appear inherently nonlocal in the relativistic limit and therefore violate causality.

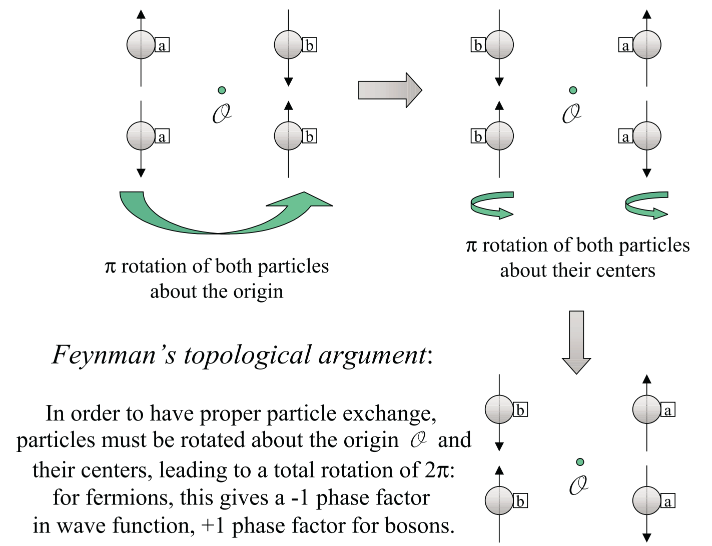

Figure 1: Intuitive argument developed by Feynman to explain the spin-statistics connection. Here we imagine the particles a and b are in a coherent superposition $\psi(a,b) = \frac{1}{\sqrt{2}}(|\uparrow\rangle_a|\downarrow\rangle_b \pm |\uparrow\rangle_b|\downarrow\rangle_a)$. The basic problem with the argument is the "markers" on the particles (telling us which way they face). There is no fundamental justification for why the final rotation is necessary. The markers can be replaced with hypothetical tethers between the particles or more formal mathematics (spin basis), but the same objection applies.

A crucial theoretical question is whether or not the spin-statistics connection can be derived by simpler methods, in particular with nonrelativistic quantum mechanics. A simpler proof might be able to connect spin-statistics with more well-established assumptions of quantum mechanics and allow strict limits to be placed on possible violations. An excellent summary of such attempts and their associated problems, as well as a simplified proof using relativistic quantum mechanics, is presented in [I. Duck and E.C.G. Sudarshan, Amer. J. of Phys. **66**, 284 (1998)]. One of the most notable arguments for the spin-statistics theorem is made by Feynman [R.P. Feynman, in: *The 1986 Dirac Memorial Lectures*, (Cambridge University Press, Cambridge, 1987)], and is summarized in Fig. 1. The argument, which involves topological markers, has recently been formally extended by Berry and Robbins [M.V. Berry and J.M. Robbins, Proc. Roy. Soc. London A **453**, 1771 (1997)]. The crucial problem with these rather intuitive arguments is that they add some new postulate to quantum mechanics whose existence serves solely to prove the spin-statistics theorem. Therefore they cannot provide a more strict limit on violations of the spin-statistics theorem, and in fact appear redundant since relativistic proofs already exist. These nonrelativistic proofs also imply that the spatial-spin part of any wavefunction is symmetric for integer-spin particles and antisymmetric for half-integer spin particles. This conclusion is not strictly true in general, however, since the wavefunction can possess additional degrees of freedom (such as color) [J. Anandan, Phys. Lett. A **248**, 124 (1998)]. The symmetrization postulate implies that the additional degrees of freedom must assume symmetric or antisymmetric states (unlike nonidentical particles, which need not assume any particular exchange symmetry).

In considering the possibility of a small violation of Bose-Einstein statistics for photons, it is necessary to ask how such a violation could manifest itself? We can assume the validity of nonrelativistic quantum mechanics in all our arguments, since spin-statistics violations do not appear to contradict any assumptions of this theory (although the description of creation and annhilation of photons without reference to relativistic quantum theory is a separate story). The spin-statistics theorem concerns identical particles, which means that the particles cannot be distinguished by any measurement. In particular, this means that all operators corresponding to physical observables must commute with all exchange operators ξ, for example $[H,\xi]=0$ for any Hamiltonian. This fact is used by Amado and Primakoff to derive a *superselection rule*: the symmetry class of a state is conserved and different symmetry classes do not mix [R. Amado and H. Primakoff, Phys. Rev. C **22**, 1338 (1980)]. This implies:

$$\langle A|H|S\rangle = 0, \tag{5}$$

where H is any Hamiltonian, and $|S\rangle$ and $|A\rangle$ are symmetric and antisymmetric states respectively. In other words, if no new particles were created, then a class of particles which started out in a symmetric state would remain symmetric for all time. As Feynman said, "so the problem would be pushed back to Creation itself, and God only knows how that was done." However, it is important to note that the creation and annihilation of particles is beyond the scope of nonrelativistic quantum mechanics. The quon algebra, for example, takes advantage of this exception by postulating creation and annihilation operators which do not obey the usual commutation relations, leading to the creation of particle states which are neither symmetric nor antisymmetric.

In the following discussion we assume that a small violation of the spin-statistics and/or symmetrization postulate would manifest itself as an intrinsic property of all photons. One way to characterize this violation is to place all photons in a common symmetry class which is neither symmetric nor antisymmetric, but rather is described by a formal superposition of these two states Ψ:

$$\Psi = \sqrt{1-\nu}|S\rangle \pm \sqrt{\nu}|A\rangle, \tag{6}$$

where $|S\rangle$ is the symmetric wave function, $|A\rangle$ is the antisymmetric wave function, and ν is the probability for any process occurring only for an antisymmetric photon state. When we say the photon state is described by a formal superposition, we mean that even though $[H,\xi]=0$ implies that we can write energy eigenstates of our Hamiltonian (and, for that matter, any other operator) as eigenstates of the exchange operator, we choose instead to write them as the above superposition. We assume there is some as yet undiscovered principle which forces us to make this choice. This is equivalent to a violation of the *symmetrization postulate*, which states that all wave functions are eigenfunctions of the exchange operator.

The fact that the above state (Eq. (6)) violates the symmetrization postulate can be illustrated with a generic two photon state $\Psi(a,b)$. If two photons are interchanged, the expectation value of any physical observable is unaffected since photons are identical. This implies, since $\Psi^*\Psi$ is an observable, that if $\Psi(a,b)$ is an eigenfunction of the exchange operator ξ_{ab}:

$$\xi_{ab}\Psi(a,b) = \Psi(b,a) = e^{i\phi}\Psi(a,b),$$

where ϕ is some phase. We can solve for ϕ by performing an additional interchange to return us to our original state:

$$\xi_{ab}^2\Psi(a,b) = \xi_{ab}e^{i\phi}\Psi(b,a) = e^{2i\phi}\Psi(a,b),$$

so $\phi = 0,\pi$ (i.e., $e^{i\phi}=\pm 1$). So the symmetrization postulate explicitly forbids particles from being in a superposition of symmetric and antisymmetric states.

The state Ψ behaves as an incoherent mixture precisely because the Primakoff superselection rule forbids cross terms between states of different permutation symmetry in the expectation value of any operator Θ:

$$\langle\Psi|\Theta|\Psi\rangle = (1-\nu)\langle S|\Theta|S\rangle + \nu\langle A|\Theta|A\rangle. \tag{7}$$

However, in my opinion, the above description of spin-statistics violation is a little different from describing the photons with a density matrix:

$$\begin{pmatrix} \rho_S & 0 \\ 0 & \rho_A \end{pmatrix} = \begin{pmatrix} 1-\nu & 0 \\ 0 & \nu \end{pmatrix}, \tag{8}$$

where ρ_S and ρ_A are the populations of the exchange-symmetric and exchange-antisymmetric states, respectively. The primary difference is that in the density matrix, ν of the photons are antisymmetric with respect to interchange while they have no definite symmetry with respect to the other photons. When the photons are described by a formal superposition of symmetric and antisymmetric states, there is a probability ν for any two photons to be antisymmetric with respect to interchange and no possibility that their exchange symmetry is undefined. For this reason, I prefer to use the formal superposition.

To understand what is meant by a "symmetry class," let's consider some interaction which generates an energy splitting $\hbar\omega_{as}$ between $|S\rangle$ and $|A\rangle$. This interaction introduces a time-dependent phase between $|S\rangle$ and $|A\rangle$, but in fact won't change the symmetry class in accordance with the Primakoff superselection rule:

$$\Psi(t) = \sqrt{1-\nu}|S\rangle \pm e^{i\omega_{as}t}\sqrt{\nu}|A\rangle. \tag{9}$$

The accrued phase cancels in evaluation of any expectation value for $\Psi(t)$, and this is what is meant by symmetry class – the particular ratio between symmetric and antisymmetric contributions to the wave function is fixed for all members of a given symmetry class. It is important to note that because of the Primakoff superselection rule, there cannot be transitions $\Psi \to \Psi'$ where Ψ' possesses a different symmetry.

Two photon states and the Landau-Yang Theorem

Suppose we have two photons (a and b) in a state with a spatially symmetric wave function. The possible spin states of the photons can be found starting from the stretched state and applying lowering operators. The $J_{tot} = 2$ multiplet consists of symmetric spin functions (where the states are labeled by $|J, M\rangle$, where J is the total angular momentum of a particle or the system and M is the projection along a particular axis):

$$|2,2\rangle_{tot} = |1\rangle_a |1\rangle_b,$$

$$|2,1\rangle_{tot} = \frac{1}{\sqrt{2}}(|1\rangle_a|0\rangle_b + |0\rangle_a|1\rangle_b),$$

$$|2,0\rangle_{tot} = \frac{1}{\sqrt{6}}(|1\rangle_a|{-1}\rangle_b + 2|0\rangle_a|0\rangle_b + |{-1}\rangle_a|1\rangle_b),$$

$$|2,-1\rangle_{tot} = \frac{1}{\sqrt{2}}(|0\rangle_a|{-1}\rangle_b + |{-1}\rangle_a|0\rangle_b),$$

and

$$|2,-2\rangle_{tot} = |{-1}\rangle_a|{-1}\rangle_b.$$

The $J_{tot} = 1$ multiplet consists of anitsymmetric spin functions, and is therefore forbidden by the spin-statistics theorem:

$$|1,1\rangle_{tot} = \frac{1}{\sqrt{2}}(|1\rangle_a|0\rangle_b - |0\rangle_a|1\rangle_b),$$

$$|1,0\rangle_{tot} = \frac{1}{\sqrt{2}}(|1\rangle_a|{-1}\rangle_b - |{-1}\rangle_a|1\rangle_b),$$

and

$$|1,-1\rangle_{tot} = \frac{1}{\sqrt{2}}(|0\rangle_a|{-1}\rangle_b - |{-1}\rangle_a|0\rangle_b).$$

Finally, there is the symmetric $|0,0\rangle_{tot} = |0\rangle_a|0\rangle_b$ state.

An immediate consequence is that a vector ($J = 1$) particle cannot decay into 2 photons. Since the photons conserve linear momentum in the particle rest frame and space is isotropic, they must be emitted in spherical waves, so the spatial part of the wavefunction is symmetric. In order to conserve angular momentum, the photons must be in a $J_{tot} = 1$ state – however this state is antisymmetric and therefore forbidden. This is known as the Landau-Yang theorem [L.D. Landau, Dokl. Akad. Nauk., USSR **60**, 207 (1948); C.N. Yang, Phys. Rev. **77**, 242 (1950)]. If Bose-Einstein statistics for photons is violated, there can arise a branching ratio $\propto \nu$ for the forbidden decay into two photons. The decay of the Z_0-boson into two photons is forbidden by the Landau-Yang theorem, but the current limit on ν from Z_0 decay data is very poor, $\nu < 1$ [A. Yu. Ignatiev, G.C. Joshi, and M. Matsuda, Mod. Phys. Lett. A **11**, 871 (1996)].

Recently, a direct atomic physics test of Bose-Einstein statistics based on this principle has been conducted [D. DeMille, D. Budker, N. Derr, and E. Deveney, Phys. Rev. Lett. **83**, 3978 (1999)]. This experiment searched for a two-photon $J = 0 \to J' = 1$ transition with degenerate photons in atomic barium. The result sets a limit $\nu < 1.2 \times 10^{-7}$. A new experiment, aimed at improving this limit by several orders of magnitude is in progress here at Berkeley (headed up by our very own Damon Brown and Dima Budker).

N photon states and Statistical Ensembles

There is a great deal of debate in the literature concerning spin-statistics violations about how to represent multi-particle states. Since I am presently unable to come to a solid conclusion, I propose to interpret a violation of the symmetrization postulate and the spin statistics theorem to imply that all photons are in a mixed permutation symmetry group described by wave functions of the form

$$\Psi = \sqrt{1-\nu}|S\rangle + \sqrt{\nu}|A\rangle,$$

where ν characterizes the probability of finding two photons in an exchange-antisymmetric state. This would suggest that no matter how many photons you have, ν part of the time you measure them to be in exchange-antisymmetric states.

Large statistical ensembles of fermions and bosons can in principle be distinguished by their distribution functions $f_{\rm FD}$ and $f_{\rm BE}$, respectively, where for photons (chemical potential = 0):

$$f_{\rm FD} = \frac{1}{e^{\epsilon_i/k_BT} + 1} \tag{10}$$

and

$$f_{\rm BE} = \frac{1}{e^{\epsilon_i/k_BT} - 1}, \tag{11}$$

where ϵ_i is the energy of a state i. The distributions give the average number of particles in state i for a thermal distribution. I believe, although it is a bit difficult to prove presently, that the distribution function $f_{\rm BV}$ for photons including a small violation of the spin-statistics theorem would then be given by:

$$f_{\rm BV} = (1-\nu)f_{\rm BE} + \nu f_{\rm FD}. \tag{12}$$

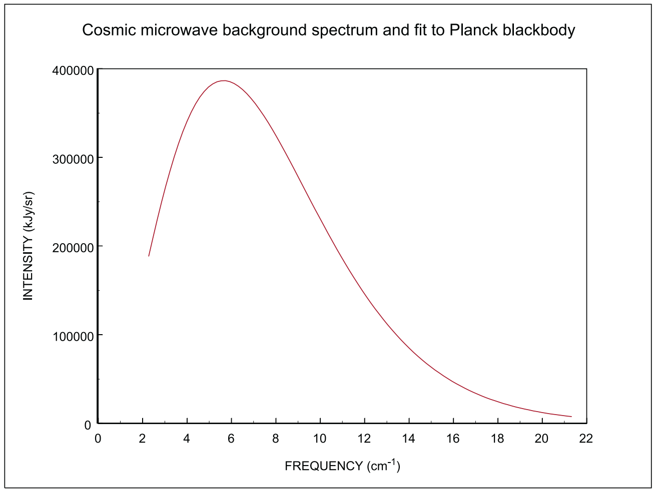

Figure 2: Cosmic background radiation spectrum and fit to Planck blackbody spectrum. Uncertainties are a small fraction of the line thickness.

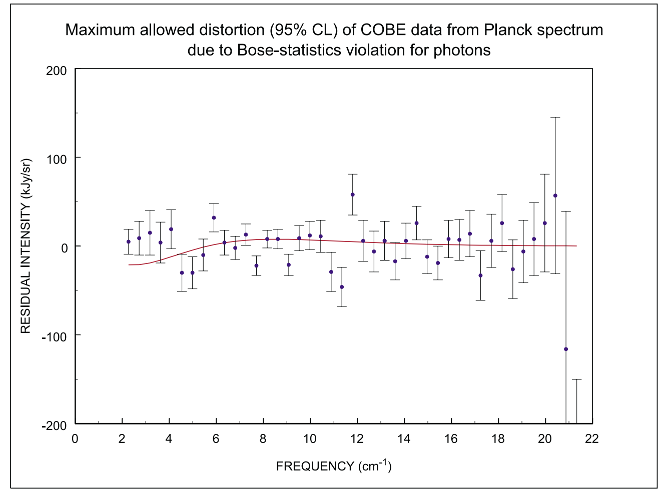

Figure 3: Residuals of the the fit of the cosmic microwave background radiation to a Planck spectrum and maximum allowed distortion (95% CL) due to violation of the spin-statistics theorem for photons.

Blackbody Spectrum

From the above (slightly tenuous) analysis, the Bose-violating Planck spectrum is described by:

$$\rho(\omega,T)=\frac{\hbar c^2\omega^3}{\pi^2}\left(\frac{1-\nu}{e^{(\hbar\omega/k_BT)}-1}+\frac{\nu}{e^{(\hbar\omega/k_BT)}+1}\right), \quad (13)$$

where $\rho(\omega,T)$ is the energy density per unit frequency and ω is the frequency.

One of the most accurate measurements of a blackbody spectrum is that of the cosmic microwave background (CMB) radiation. Figure 2 shows the results of measurements of the spectrum of the CMB radiation by the Far-InfraRed Absolute Spectrophotometer (FIRAS) on board the COsmic Microwave Background Explorer (COBE) satellite [see, e.g., D.J. Fixsen, et.al., Astr. Phys. Journal ***f***473, 576 (1996)]. The deviations of the data from a perfect Planck spectrum are smaller than the width of the line in the plot. The residuals from the fit of the COBE data to a Planck blackbody spectrum can be used to obtain a limit on ν.

Simulated data indicates that the residual difference between a Bose-violating spectrum and the fitted Planck spectrum are well described by the function:

$$\delta\rho(\omega)=\nu\cdot\frac{\hbar c^2\omega^3}{\pi^2}\left(\frac{1}{e^{(\hbar\omega/k_BT)}+1}-\frac{\varrho}{e^{(\hbar\omega/k_BT)}-1}\right), \quad (14)$$

where $\delta\rho(\omega)$ is the residual difference between the Bose-violating spectrum and the fitted Planck spectrum and $\varrho\sim 1$ is a parameter related to the fitting (since a better fit is obtained by slightly overestimating the Bose-conserving contribution to the spectrum). We use Eq. (14) to estimate the maximum allowed distortion from the Planck spectrum from the residuals. The residuals are fit to Eq. (14) with ϱ as a free parameter, and the reduced χ^2 is inflated to a level where the likelihood of the fit being correct is only 5%. From this analysis we obtain a limit on ν. Figure 3 shows the maximum allowed distortion fit and the residuals from the COBE data. From the fit, we find $\nu<3\times10^{-4}$ at the 95% confidence level. This can be compared to the limit obtained from the experiment by DeMille et. al. of $\nu<1.2\times10^{-7}$ at the 95% confidence level [D. DeMille, D. Budker, N. Derr, and E. Deveney, Phys. Rev. Lett. **83**, 3978 (1999)]. However, it should be noted that there is a very different energy range and method of photon creation for the two measurements.

Lasers

There is also evidence from the existence of lasers that photons obey Bose-statistics. A large number of photons in the same mode of the electromagnetic field means that a large number of photons occupy an exchange-symmetric state. There was attempt to derive a limit on Bose-statistics violations for photons from the existence of high powered lasers based on the quon theory [D.I. Fivel, Phys.

Rev. A **f**43, 4913 (1991)], but some flaws were discovered in this argument [O.W. Greenberg, in *Workshop on Harmonic Oscillators*, NASA Conference Pub. 3197, eds. D. Han, Y.S. Kim, and W.W. Zachary (NASA, Greenbelt, 1993)]. Without a specific model, it appears difficult to place a limit on ν from laser behavior, since the population of the exchange-antisymmetric mode would always be a small fraction of the population of the symmetric mode. Roughly, the effect would manifest itself as a linearly increasing (with respect to laser power) occupation of another mode (the number of photons in the antisymmetric mode $N_A \sim \nu N_S$, where N_S is the number of photons in the usual symmetric mode). To my knowledge, no sensitive search of this kind has been performed. However, the recent experiment of DeMille et. al. seems to be a good way to measure the "fraction" of photons emitted from a specific laser which don't obey Bose statistics.

Problem 11 *Power spectrum of exponentially growing/decaying harmonic field*

The Fourier transform of an exponentially decaying harmonically oscillating field is a Lorentzian:

$$f_-(\omega) = \int_0^\infty e^{-t\gamma/2}\sin(\Omega t)e^{-i\omega t}dt = \frac{\Omega}{\gamma^2 + 2i\gamma\omega + (\Omega^2 - \omega^2)}, \qquad (15)$$

where γ is the decay rate and Ω is the oscillation frequency. The Fourier transform $f_+(\omega)$ of an exponentially growing harmonically oscillating field is also a Lorentzian:

$$f_+(\omega) = \int_{-\infty}^0 e^{t\gamma/2}\sin(\Omega t)e^{-i\omega t}dt = \frac{\Omega}{-\gamma^2 + 2i\gamma\omega - (\Omega^2 - \omega^2)}. \qquad (16)$$

If we add these two results, we get for the Fourier transform of exponentially growing then decaying function:

$$f_+(\omega) + f_-(\omega) = \frac{-4i\gamma\omega\Omega}{\gamma^4 + (\Omega^2 - \omega^2)^2 + 2\gamma^2(\Omega^2 + \omega^2)}. \qquad (17)$$

To obtain the power spectrum, we take the norm-square of $f_+(\omega) + f_-(\omega)$, which gives for the power spectrum:

$$|f_+(\omega) + f_-(\omega)|^2 = \frac{16\gamma^2\omega^2\Omega^2}{\left(\gamma^4 + (\Omega^2 - \omega^2)^2 + 2\gamma^2(\Omega^2 + \omega^2)\right)^2}. \qquad (18)$$

The above power spectrum is compared to the power spectrum of an exponentially decaying harmonic field in Fig. 4, which is broader than the sum of the growing and decaying fields. This difference is due to the sharp edge at $t = 0$ for the exponentially decaying field, which is known to have many Fourier components.

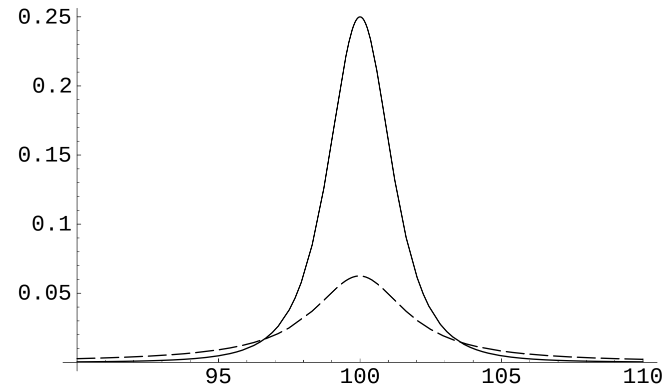

Figure 4: Solid line: power spectrum of an exponentially growing then decaying sinusoidal field ($\gamma = 2$, $\Omega = 100$); Dashed line: power spectrum of an exponentially decaying sinusoidal field for the same parameters.

The power spectrum for an exponentially decaying harmonic field is given by:

$$|f_-(\omega)|^2 = \frac{\Omega^2}{\left(\gamma^4 + (\Omega^2 - \omega^2)^2 + 2\gamma^2(\Omega^2 + \omega^2)\right)^2}. \qquad (19)$$

If the approximations $\Omega \ll \omega$ and $\omega \gg \gamma$ are made, the power spectrum for the growing and decaying harmonic field is given by:

$$|f_+(\omega) + f_-(\omega)|^2 \approx \frac{16\gamma^2\Omega^2}{\omega^2\left(2\gamma^2 + (\Omega - \omega)^2\right)^2}. \qquad (20)$$

and for the decaying harmonic field:

$$|f_+(\omega) + f_-(\omega)|^2 \approx \frac{\Omega^2}{\omega^2(2\gamma^2 + (\Omega - \omega)^2)}. \qquad (21)$$

As you can see by comparing Eqns. (20) and (21), the power spectrum of the usual Lorentzian falls off as $\frac{1}{\Delta^2}$, where $\Delta = \Omega - \omega$, whereas the power spectrum of the growing and decaying field falls off as $\frac{1}{\Delta^4}$.