

At the question period after a Dirac lecture at the University of Toronto, somebody in the audience remarked: "Professor Dirac, I do not understand how you derived the formula on the top left side of the blackboard."

"This is not a question," snapped Dirac, "it is a statement. Next question, please."

- George Gamow, excerpted from *Thirty Years that Shook Physics*.

If you have any questions, suggestions or corrections to the solutions, don't hesitate to e-mail me at dfk@uclink4.berkeley.edu!

I would also like to make a disclaimer: these solutions do not necessarily reflect the style of solution that Prof. Budker intends. Prof. Budker prefers back-of-the-envelope calculations with as little math as possible. These solutions include extra background material and derivations which motivate the simpler calculations involved in the solutions of the problems, in case some of the material is new to you.

Problem 1 *Effect of optical pumping on light absorption*

Let's begin with a quick review of the interaction of light with atoms, the results of which we can employ in the solution of the problem. The information contained in this review comes from three basic sources: I.I. Sobelman, *Atomic Spectra and Radiative Transitions*, (Springer, Berlin, 1992); B.H. Bransden and C.J. Joachain, *Physics of Atoms and Molecules*, (Longman, Essex UK, 1983); and notes from Prof. Littlejohn's Physics 138 class.

Review of the Interaction of Light with Atoms

Fermi's Golden Rule, originally obtained by Dirac from first order time-dependent perturbation theory, gives us the transition rate dW_{ab} between atomic states $|a\rangle$ and $|b\rangle$ for atoms in an electromagnetic field described by the Hamiltonian H' :

$$dW_{ab} \approx \frac{2\pi}{\hbar} |H'_{ab}|^2 \rho_a(E) = \frac{2\pi}{\hbar} |H'_{ab}|^2 \delta(E_a - E_b + \hbar\omega) \frac{V d\vec{k}}{(2\pi)^3}, \quad (1)$$

where E_a and E_b are the energies of $|a\rangle$ and $|b\rangle$, respectively, ω is the light frequency, \vec{k} is the light wave vector, and V is the volume of the box in which we confine our photons so that their wavefunctions are normalizable (it will cancel out later, of course). Equation (1) describes the emission of a photon with a wave vector in the interval \vec{k} to $\vec{k} + d\vec{k}$ and polarization $\hat{\epsilon}_{\vec{k},j} \perp \vec{k}$. The perturbing Hamiltonian (for a single electron atom, although for multielectron atoms we can just include a sum over all the electrons) is given by:

$$H' = -\frac{e}{mc} \vec{p} \cdot \hat{\epsilon}_{\vec{k},j} \left(a_{\vec{k},j} e^{i\vec{k} \cdot \vec{r}} + a_{\vec{k},j}^\dagger e^{-i\vec{k} \cdot \vec{r}} \right), \quad (2)$$

where \vec{p} is the electron momentum, $a_{\vec{k},j}$ and $a_{\vec{k},j}^\dagger$ are the appropriate photon annihilation and creation operators. This Hamiltonian comes from including the vector potential $\vec{A}(\vec{r}, t)$ due to light in the Hamiltonian describing our atomic system:

$$H = \frac{1}{2m} \left(\vec{p} + \frac{e}{c} \vec{A}(\vec{r}, t) \right)^2 + V(r).$$

If we expand the kinetic energy term in the Hamiltonian, we find that:

$$T = \frac{p^2}{2m} + \frac{e}{2mc} \left(\vec{p} \cdot \vec{A} + \vec{A} \cdot \vec{p} \right) + \frac{e^2}{2mc^2} \vec{A}^2.$$

The two cross terms are equal in the Coulomb gauge and the term $\frac{e^2}{2mc^2} \vec{A}^2$ is assumed small since \vec{A} is small for the light fields in this problem, so we have:

$$H = \frac{p^2}{2m} + V(r) + \frac{e}{mc} \vec{p} \cdot \vec{A}.$$

Quantization of the electromagnetic field (second quantization) gives us the form of the perturbing Hamiltonian in Eqn. (2).

We note that $\langle a | H' | b \rangle$ is nonzero only when the number of photons in the \vec{k}, j mode increases or decreases by unity. Thus we have for emission:

$$\langle a, n+1 | H' | b, n \rangle = -\frac{e}{mc} \sqrt{\frac{2\pi\hbar c^2(n+1)}{\omega V}} \hat{\epsilon}_{\vec{k},j} \cdot \langle a | \vec{p} e^{i\vec{k} \cdot \vec{r}} | b \rangle, \quad (3)$$

and for absorption:

$$\langle b, n-1 | H' | a, n \rangle = -\frac{e}{mc} \sqrt{\frac{2\pi\hbar c^2 n}{\omega V}} \hat{\epsilon}_{\vec{k},j} \cdot \langle b | \vec{p} e^{-i\vec{k} \cdot \vec{r}} | a \rangle, \quad (4)$$

where n is the number of photons in a particular mode. Employing Eqns. (3) and (4) in Fermi's Golden Rule, Eqn. (1), we obtain:

$$\frac{dW_{ab}}{d\Omega} = \frac{e^2 \omega}{2\pi\hbar c^3 m^2} |\hat{\epsilon}_{\vec{k},j} \cdot \langle a | \vec{p} e^{i\vec{k} \cdot \vec{r}} | b \rangle|^2 (\bar{n}_{\vec{k},j} + 1) \quad (5)$$

$$\frac{dW_{ba}}{d\Omega} = \frac{e^2\omega}{2\pi\hbar c^3 m^2} |\hat{\epsilon}_{\hat{k},j} \cdot \langle b|\vec{p}e^{-i\vec{k}\cdot\vec{r}}|a\rangle|^2 \left(\bar{n}_{\hat{k},j}\right), \quad (6)$$

where $\bar{n}_{\hat{k},j}$ is the mean number of photons in a particular mode. Equation (5) describes spontaneous and stimulated transitions from $|b\rangle \rightarrow |a\rangle$ and Eqn. (6) describes stimulated transitions from $|a\rangle \rightarrow |b\rangle$.

We employ the electric dipole (E1) approximation to describe the interaction of the atom with the light field. Basically, we assume that $\lambda \gg d$ where λ is the wavelength of the light and d is the characteristic dimension of the atom. This allows us to assume

$$\vec{k} \cdot \vec{r} \ll 1,$$

and thus $e^{i\vec{k}\cdot\vec{r}} \sim 1$. So we may write

$$\langle b|\vec{p}e^{i\vec{k}\cdot\vec{r}}|a\rangle \approx \langle b|\vec{p}|a\rangle. \quad (7)$$

We can invoke a simple commutation relation, namely:

$$[\vec{r}, H_0] = i\hbar \frac{d\vec{r}}{dt} = \frac{i\hbar\vec{p}}{m},$$

where H_0 is the atomic Hamiltonian, to find that:

$$\langle b|\vec{p}|a\rangle = \frac{-im}{\hbar} \langle b|(\vec{r}H_0 - H_0\vec{r})|a\rangle = -im\omega \langle b|\vec{r}|a\rangle, \quad (8)$$

Now we introduce the transition electric dipole moment,

$$\vec{d}_{ba} = \langle b|e\vec{r}|a\rangle,$$

and we can write the transition rates from Eqns. (5) and (6) in terms of \vec{d}_{ba} :

$$\frac{dW_{ab}}{d\Omega} = \frac{\omega^3}{2\pi\hbar c^3} |\hat{\epsilon}_{\hat{k},j} \cdot \vec{d}_{ab}|^2 \left(\bar{n}_{\hat{k},j} + 1\right), \quad (9)$$

$$\frac{dW_{ba}}{d\Omega} = \frac{\omega^3}{2\pi\hbar c^3} |\hat{\epsilon}_{\hat{k},j} \cdot \vec{d}_{ba}|^2 \left(\bar{n}_{\hat{k},j}\right). \quad (10)$$

In this problem, we choose the \hat{z} -axis along \vec{d} . We now want to investigate how the transition rates depend on the angular momenta of the upper and lower states. Let

us re-label $|a\rangle = |\gamma JM\rangle$ and $|b\rangle = |\gamma' J' M'\rangle$, where J, J' are total angular momenta of the lower and upper states, M, M' are the projections of the angular momentum along the \hat{z} -axis and γ, γ' account for any additional quantum numbers. Then we can decompose $\hat{\epsilon}_{\hat{k},j} \cdot \vec{d}_{ab}$ using the spherical basis:

$$\hat{e}_1 = -\frac{1}{\sqrt{2}}(\hat{x} + i\hat{y}),$$

$$\hat{e}_0 = \hat{z},$$

$$\hat{e}_{-1} = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{y}).$$

So we have

$$\hat{\epsilon}_{\hat{k},j} \cdot \vec{d}_{ab} = \sum_q \epsilon_q \langle \gamma JM | d_q | \gamma' J' M' \rangle \quad (11)$$

Now we can invoke the Wigner-Eckart theorem for irreducible tensor operators (T_q^k), which states:

$$\langle \gamma JM | T_q^k | \gamma' J' M' \rangle = \langle \gamma J || T^k || \gamma' J' \rangle \langle JM | J' k M' q \rangle. \quad (12)$$

where $\langle \gamma J || T^k || \gamma' J' \rangle$ is the reduced matrix element and $\langle JM | J' k M' q \rangle$ are the appropriate Clebsch-Gordan coefficients. Please note that there are other definitions of the reduced matrix element floating about in the literature, in particular Sobelman uses:

$$\langle \gamma JM | T_q^k | \gamma' J' M' \rangle = \frac{1}{\sqrt{1+2k}} \langle \gamma J || T^k || \gamma' J' \rangle \langle JM | J' k M' q \rangle.$$

The dipole operator is a rank one tensor ($k = 1$). The pump light is polarized in the \hat{z} -direction, so for optical pumping we only need to consider the $q = 0$ case, but for spontaneous emission any polarization is allowed, so we must include $q = -1, 0, 1$ cases.

So for absorption we have:

$$\frac{dW_{ba}}{d\Omega} = \frac{\omega^3 \bar{n}}{2\pi\hbar c^3} |\langle \gamma' J' || d || \gamma J \rangle|^2 |\langle J' M' | J 1 M 0 \rangle|^2. \quad (13)$$

and for spontaneous emission to a particular Zeeman sublevel $|JM\rangle$ (here we neglect stimulated emission), we have:

$$\frac{dW_{ab}}{d\Omega} = \frac{\omega^3}{2\pi\hbar c^3} \sum_{q,M'} |\langle \gamma J || d || \gamma' J' \rangle|^2 |\langle JM | J' 1 M' q \rangle|^2. \quad (14)$$

1 → 0 Case

We begin with an isotropic distribution of atoms in the ground state Zeeman sublevels ($N/3$ in each sublevel, where N is the total number of atoms). Note that in the low light intensity regime, the attenuation of the light intensity is proportional to the number of atoms in ground state sublevels which interact with light times the probability for absorption from such a sublevel. Only one of the ground state Zeeman sublevels, namely $M = 0$, interacts with the pump light in the $1 \rightarrow 0$ case. Some fraction ΔN of the atoms, which depends on the pump light intensity, will be excited to the upper state.

The $|00\rangle$ state can decay into any of the ground state sublevels. From Eqn. (14) we see that the only difference in the formulae for decay rates to different sublevels is the Clebsch-Gordan coefficients involved. For the $0 \rightarrow 1$ transitions, the spontaneous decay rates are proportional to:

$$\sum_{q,M'} |\langle 11 | J' 1 M' q \rangle|^2 = |\langle 11 | 0101 \rangle|^2 = 1/3,$$

$$\sum_{q,M'} |\langle 10 | J' 1 M' q \rangle|^2 = |\langle 10 | 0100 \rangle|^2 = 1/3,$$

$$\sum_{q,M'} |\langle 1-1 | J' 1 M' q \rangle|^2 = |\langle 1-1 | 010-1 \rangle|^2 = 1/3.$$

So in fact the atoms excited to the upper state decay with equal likelihood to any of the ground state sublevels. This can also be deduced just from the isotropy of space, since atoms in the $|00\rangle$ state are unpolarized, so they must decay with equal likelihood to each of the ground state sublevels. Thus we conclude the ground state populations after one optical pumping cycle are given by:

$$N_1 = \frac{N}{3} + \frac{\Delta N}{3}$$

$$N_0 = \frac{N}{3} - \frac{2\Delta N}{3}$$

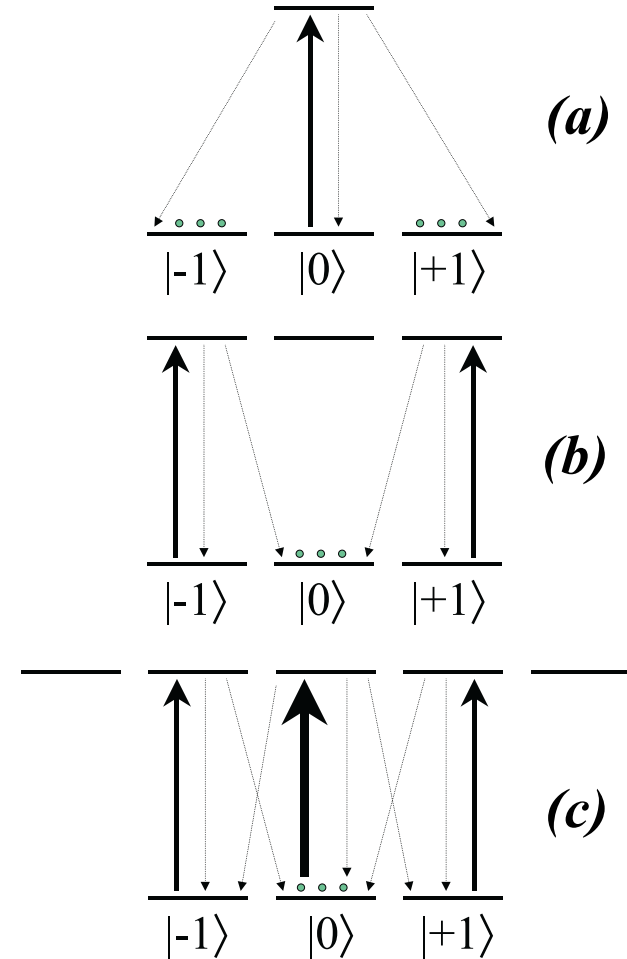


Figure 1: Illustration of the effects of optical pumping on an $J = 1$ ground state for $J \rightarrow J - 1$, J , and $J + 1$ transitions. The incident light is linearly polarized in the \hat{z} direction. We choose our quantization axis along the \hat{z} direction. (a) $1 \rightarrow 0$ transition: atoms are pumped into the $| +1 \rangle$ and $| -1 \rangle$ states, both of which are dark, so optical pumping decreases absorption. (b) $1 \rightarrow 1$ transition: atoms are pumped into $| 0 \rangle$, which is a dark state. (c) $1 \rightarrow 2$ transition: atoms tend to be pumped into $| 0 \rangle$, which interacts more strongly with the light field (as can be verified by comparison of Clebsch-Gordan coefficients). Thus optical pumping increases absorption.

$$N_{-1} = \frac{N}{3} + \frac{\Delta N}{3}$$

So the population of the dark states (states which do not interact with the pump light) increases, while the population of the bright state $|10\rangle$ decreases. This causes absorption of light to decrease.

1 → 1 Case

In this case two ground state sublevels ($|11\rangle, |1-1\rangle$) interact with the pump light while $|10\rangle$ is a dark state. This can be seen from Eqn. (13) by comparing the Clebsch-Gordan coefficients:

$$|\langle J'M'|J1M0\rangle|^2 = |\langle 11|1110\rangle|^2 = 1/2,$$

$$|\langle J'M'|J1M0\rangle|^2 = |\langle 10|1100\rangle|^2 = 0,$$

$$|\langle J'M'|J1M0\rangle|^2 = |\langle 1-1|11-10\rangle|^2 = 1/2.$$

So the transition rates from $|11\rangle, |1-1\rangle$ are the same. Once again, we will assume that ΔN atoms from each of the bright states make transitions to the appropriate excited states.

Decay from the the excited Zeeman sublevels to the respective ground state sublevels is described by Eqn. (14), and the relative rates are proportional to:

$$\sum_{q,M'} |\langle 11|J'1M'q\rangle|^2 = |\langle 11|1101\rangle|^2 = 1/2,$$

$$\sum_{q,M'} |\langle 10|J'1M'q\rangle|^2 = |\langle 10|111-1\rangle|^2 + |\langle 10|11-11\rangle|^2 = 1/2 + 1/2 = 1,$$

$$\sum_{q,M'} |\langle 1-1|J'1M'q\rangle|^2 = |\langle 1-1|11-10\rangle|^2 = 1/2.$$

From which we can deduce the relative changes in population of the ground state Zeeman sublevels:

$$N_1 = \frac{N}{3} - \frac{\Delta N}{2}$$

$$N_0 = \frac{N}{3} + \Delta N$$

$$N_{-1} = \frac{N}{3} - \frac{\Delta N}{2}$$

The bright states are depleted and atoms are pumped into the dark state, so the absorption of light decreases.

1 → 2 Case

In this case, all three ground state Zeeman sublevels interact with the pump light, i.e. there are no dark states. The strength of the interaction with light (absorption rate) does vary between the states, as can be seen by comparing the Clebsch-Gordan coefficients.

$$|\langle J'M'|J1M0\rangle|^2 = |\langle 21|1110\rangle|^2 = 1/2,$$

$$|\langle J'M'|J1M0\rangle|^2 = |\langle 20|1100\rangle|^2 = 2/3,$$

$$|\langle J'M'|J1M0\rangle|^2 = |\langle 2-1|11-10\rangle|^2 = 1/2.$$

In this situation, ΔN atoms are pumped from the $|11\rangle, |1, -1\rangle$ states and $(4/3) \cdot \Delta N$ atoms are pumped from the $|10\rangle$ state. Now we can compute the relative rates of spontaneous decay into the various sublevels, taking into account the difference in initial population of the excited states. Also to get the right total number of atoms, we have to normalize the rates from a given excited state sublevel. For the $1 \rightarrow 0, 1$ cases, the Clebsch-Gordan coefficients worked out so that they were already normalized as you can verify.

First, let's take care of the normalization. The possible decays from $|21\rangle$ are described by the Clebsch-Gordan coefficients:

$$|\langle 11|2110\rangle|^2 = \frac{3}{10}$$

$$|\langle 10|211-1\rangle|^2 = \frac{3}{10}.$$

So half the atoms from $|21\rangle$ decay to $|11\rangle$ and half decay to $|10\rangle$. Analogously, half of the atoms from $|2-1\rangle$ decay to $|1-1\rangle$ and half decay to $|10\rangle$. For the $|20\rangle$ state, we find that:

$$|\langle 11|2101\rangle|^2 = \frac{1}{10},$$

$$|\langle 10|2100\rangle|^2 = \frac{2}{5},$$

$$|\langle 1-1|210-1\rangle|^2 = \frac{1}{10}.$$

So we find that 2/3 of the atoms from the $|20\rangle$ state end up in the $|10\rangle$ state, 1/6 decay to $|11\rangle$ and 1/6 decay to $|1-1\rangle$. To normalize all the coefficients, we just need to multiply by 5/3. This turns out to be related to the reduced matrix

element which we have ignored. Taking into account the initial populations of the excited state Zeeman sublevels, we find:

$$\sum_{q,M'} |\langle 11 | J' 1 M' q \rangle|^2 = \frac{4}{3} |\langle 11 | 2101 \rangle|^2 + |\langle 11 | 2110 \rangle|^2 = \frac{2}{15} + \frac{3}{10} = \frac{13}{30}$$

$$\begin{aligned} \sum_{q,M'} |\langle 10 | J' 1 M' q \rangle|^2 &= |\langle 10 | 211 - 1 \rangle|^2 + |\langle 10 | 21 - 11 \rangle|^2 + \frac{4}{3} |\langle 10 | 2100 \rangle|^2 \\ &= \frac{3}{10} + \frac{3}{10} + \frac{8}{15} = \frac{34}{30} \end{aligned}$$

$$\sum_{q,M'} |\langle 1 - 1 | J' 1 M' q \rangle|^2 = |\langle 1 - 1 | 21 - 10 \rangle|^2 + \frac{4}{3} |\langle 1 - 1 | 210 - 1 \rangle|^2 = \frac{2}{15} + \frac{3}{10} = \frac{13}{30}$$

Then multiplying by 5/3 gives us the amount of atoms which decay back into each ground state sublevel. Now we can easily compute the change in population for each of the sublevels:

$$N_1 = \frac{N}{3} - \Delta N + \frac{13\Delta N}{18} = \frac{N}{3} - \frac{5\Delta N}{18}$$

$$N_0 = \frac{N}{3} - \frac{4\Delta N}{3} + \frac{34\Delta N}{18} = \frac{N}{3} + \frac{5\Delta N}{9}$$

$$N_{-1} = \frac{N}{3} - \Delta N + \frac{13\Delta N}{18} = \frac{N}{3} - \frac{5\Delta N}{18}$$

So we see that the population of the $|10\rangle$ state, which is more absorbing, increases. Therefore absorption of light increases in this case!

Problem 2 *Magnetic field inside a magnetized sphere*

First, we need to determine the magnetic field inside a uniformly magnetized sphere. We can think of this problem in terms of the bound surface and volume currents for such a system (pardon me for using SI units here, we'll convert to CGS in the end for those who prefer the gentler units), given by [see, e.g., D. Griffiths, *Introduction to Electrodynamics*, (Prentice-Hall, Upper Saddle River, 1999)]:

$$\vec{J}_b = \nabla \times \vec{M} = 0, \tag{15}$$

$$\vec{K}_b = \vec{M} \times \hat{n} = M \sin \theta \hat{\phi}. \tag{16}$$

Let's determine the vector potential \vec{A} due to the bound surface current.

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r}')}{\rho} da = \frac{\mu_0}{4\pi} \int \frac{M \sin \theta \hat{\phi}}{\rho} da, \tag{17}$$

where $da = R^2 \sin \theta d\theta d\phi$ and $\rho = \sqrt{R^2 + r^2 - 2Rr \cos \theta}$ is the distance from the point of interest to da . This result of the integral for points inside the sphere ($r < R$) is given by:

$$\vec{A}(r, \theta, \phi) = \frac{\mu_0 M}{3} r \sin \theta \hat{\phi}. \tag{18}$$

The magnetic field inside a uniformly magnetized sphere then follows from $\vec{B} = \nabla \times \vec{A}$:

$$\vec{B} = \nabla \times \vec{A}(r, \theta, \phi) = \frac{2\mu_0 M}{3} \hat{z}. \tag{19}$$

In CGS units, our result is:

$$\vec{B} = \frac{8\pi M}{3} \hat{z}. \tag{20}$$

The magnetic field is uniform and independent of the radius of the sphere. So the field inside a small imaginary sphere within the large sphere is in this sense entirely due to itself. If you carved out a small sphere inside the larger one (by superimposing a small sphere of opposite magnetization), we would see that the field inside the cavity is zero.

Therefore, the magnetic field which the little sphere sees is:

$$\vec{B}(\text{small sphere}) = 0$$

This is good for optical pumping magnetometers because it allows them to sample the field due to external sources rather than being sensitive to the field produced when the atoms are polarized. Note that this result does not hold for non-spherical cells!

Problem 3 *Measurement of electric fields with the quadratic Stark Shift*

The Stark shift is proportional to the square of the electric field:

$$\Delta_S = \alpha E^2 \quad (21)$$

we are given that for $E = 10$ kV/cm, the sensitivity to Stark shifts is

$$\frac{\delta\Delta_S}{\Delta_S} = 10^{-4}.$$

(a)

The relationship between the uncertainty in the determination of E and the sensitivity to the Stark shift is:

$$\delta\Delta_S = \alpha 2E\delta E, \quad (22)$$

so we have

$$\frac{\delta\Delta_S}{\Delta_S} = \frac{\alpha 2E\delta E}{\alpha E^2} = 2\frac{\delta E}{E} = 10^{-4}.$$

So our uncertainty in the determination of the 10 kV/cm field is:

$$\boxed{\delta E = \frac{1}{2} \text{ V/cm}}$$

(b)

Since our sensitivity to the Stark shifts is an absolute sensitivity (doesn't vary with electric field), we have for any electric field E^* :

$$\frac{\delta\Delta_S}{\Delta_S(E^*)} = 10^{-4} \cdot \frac{(10 \text{ kV/cm})^2}{(E^*)^2}. \quad (23)$$

The smallest detectable field is when $\frac{\delta\Delta_S}{\Delta_S} = 1$. This occurs when:

$$\boxed{E^* = 100 \text{ V/cm}}$$

This result is rather interesting. The basic idea is that if we have a small field E' in addition to a larger field E_0 , the Stark shift is given by:

$$\Delta_S = \alpha(E_0 + E')^2 \approx \alpha(E_0^2 + 2E_0E').$$

So the effect of the small field is enhanced by the presence of the larger field due to interference. However, if all we have is the small field, our Stark shift is just

$$\Delta_S = \alpha(E')^2,$$

so we are less sensitive to the field. Such ideas are important in atomic parity-violation experiments, where parity-violating terms due to the weak interaction are interfered with larger terms due to the electromagnetic interaction, allowing them to be detected.

Problem 4 *Larmor precession in alkali atoms*

The shift in the energy of an atomic level due to interaction of an external magnetic field with the magnetic moments due to the spin of the electron and the orbital angular momentum is given by:

$$\Delta E = \vec{\mu} \cdot \vec{B} = \hbar\Omega_L, \quad (24)$$

where Ω_L is the Larmor frequency and the magnetic moment of the particular atomic state is given by:

$$\vec{\mu} = -\mu_B(g_L\vec{L} + g_S\vec{S}), \quad (25)$$

where μ_B is the Bohr magneton, $g_L = 1$ and $g_S \approx 2$ (although not exactly, as we learned from Dirac and geonium). We can then relate the magnetic moment to the total angular momentum of the state $\vec{J} = \vec{L} + \vec{S}$:

$$\vec{\mu} = -\mu_B(\vec{L} + 2\vec{S}) = -\mu_B(\vec{J} + \vec{S}) = -\mu_B\left(\vec{J} + \frac{\langle\vec{S} \cdot \vec{J}\rangle}{J(J+1)}\vec{J}\right), \quad (26)$$

where

$$\langle\vec{S}\rangle = \frac{\langle\vec{S} \cdot \vec{J}\rangle}{J(J+1)}\vec{J}$$

gives the mean value of \vec{S} along \vec{J} times \vec{J} . We can then solve for $\langle\vec{S} \cdot \vec{J}\rangle$ in terms of eigenvalues of our system:

$$\vec{J} - \vec{S} = \vec{L},$$

$$J^2 + S^2 - 2\vec{S} \cdot \vec{J} = L^2,$$

so we find that

$$\langle \vec{S} \cdot \vec{J} \rangle = \frac{J(J+1) + S(S+1) - L(L+1)}{2}. \quad (27)$$

So the relation between the total angular momentum and the magnetic moment of the system is given by:

$$\vec{\mu} = -g_J \mu_B \vec{J}, \quad (28)$$

where

$$g_J = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}. \quad (29)$$

If we now include the nuclear spin \vec{I} in our total angular momentum $\vec{F} = \vec{I} + \vec{J}$ we have more total angular momentum (also known as mechanical angular momentum), but our interaction Hamiltonian remains the same since the nuclear magnetic moment interacts only weakly with the magnetic field. So we just change our g -factor by the projection of \vec{J} along \vec{F} . The calculation is similar to those above:

$$\vec{\mu} = -g_F \mu_B \vec{F}, \quad (30)$$

where

$$g_F = g_J \left(\frac{F(F+1) + J(J+1) - I(I+1)}{2F(F+1)} \right). \quad (31)$$

For alkali atom ground states $S=1/2$, $L=0$, $J=1/2$, and $g_J = 2$. The total angular momentum in this case can be $F = I \pm 1/2$. So the Landé factor is given by

$$g_F = \frac{F(F+1) + 3/4 - I(I+1)}{F(F+1)}.$$

For $F = I + 1/2$ we get:

$$g_F = \frac{F(F+1) + 3/4 - (F-1/2)(F+1/2)}{F(F+1)} = \frac{F+1}{F(F+1)} = \frac{1}{F} = \frac{1}{I+1/2},$$

and for $F = I - 1/2$ we get:

$$g_F = \frac{F(F+1) + 3/4 - (F+1/2)(F+3/2)}{F(F+1)} = \frac{-F}{F(F+1)} = \frac{-1}{F+1} = \frac{-1}{I+1/2}.$$

So we have for our Landé factors:

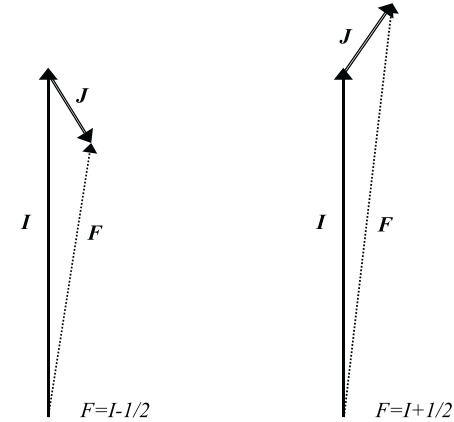


Figure 2: Illustration of the relative orientation of \vec{J} to the total angular momentum \vec{F} for $F = I \pm 1/2$.

$$g_F = \frac{\pm 1}{I + 1/2}$$

The relative sign of the two Larmor frequencies is different because \vec{J} , which is responsible for the interaction with the magnetic field, has different orientation with respect to the total angular momentum \vec{F} in the two cases. The magnitude of the Larmor frequency is reduced with increasing I due to the increase in mechanical angular momentum, since the strength of the magnetic interaction remains fixed.