Due: Tuesday, 03/29

19. Consider an atom, which has a pair of states 1 and 2 separated by an energy interval $h\nu_0$, interacting with weak monochromatic near-resonant isotropic radiation of frequency ν. According to the theory developed by Einstein, the rates of spontaneous emission, stimulated emission and absorption on the $1\to2$ transition can be described with coefficients A and B:

$$\frac{dN_{21}^{spont}}{dt} = N_2 \cdot A_{21} = N_2 \cdot 2\pi \cdot \Delta\nu_{partial},$$
$$\frac{dN_{21}^{stim}}{dt} = N_2 \cdot \int B_{21}(\nu) \cdot \rho(\nu) d\nu, \qquad (1)$$
$$\frac{dN_{12}^{absorp}}{dt} = N_1 \cdot \int B_{12}(\nu) \cdot \rho(\nu) d\nu.$$

Here we introduced the partial width of the transition $\Delta\nu_{partial}$; $\rho(\nu)=\rho\cdot\delta(\nu-\nu_0)$ is the radiation density per unit bandwidth interval ($[\rho(\nu)]=\mathrm{erg}\cdot\mathrm{cm}^{-3}\cdot\mathrm{Hz}^{-1}$). Considering thermodynamics of atoms in equilibrium with thermal radiation, Einstein found relationships between various coefficients:

$$B_{12} = \frac{g_2}{g_1} \cdot B_{21},$$
$$A_{21} = \left(\frac{8\pi \cdot h\nu^3}{c^3}\right) \cdot \int B_{21}(\nu) d\nu. \qquad (2)$$

Here $g_{1,2}$ are degeneracies of the corresponding states. These relationships allow one to derive a very general, simple and extremely useful formula for the photon absorption cross-section σ_{12}. For simplicity, first assume that the only source of line broadening is due to radiative decay of states 1 and 2.

a) Rewrite the expression for the absorption rate on the $1\to2$ transition through the photon absorption cross-section σ_{12} (σ_{12} is the cross-section averaged over light polarizations and atomic orientations).

b) Show that

$$\boxed{\sigma_{12} = 2\pi \cdot \lambda\!\!\!^{-2} \frac{\Delta\nu_{partial}}{\Delta\nu_{total}} \cdot \mathcal{L}}, \qquad (3)$$

where $\lambda\!\!\!^{-}=\lambda/2\pi$ and $\mathcal{L} = \frac{(\Delta\nu_{total})^2}{4\cdot(\nu_0-\nu)^2 + (\Delta\nu_{total})^2}$ is the normalized Lorentz line shape function.

c) Suppose 1 is the ground state and 2 is the first excited state. Estimate the typical values of σ_{12}(peak) and the integral cross-sections $\int \sigma_{12} \cdot d\nu$ if $1\to2$ is an E1, M1 and E2 transition.

d) Use the results of the previous parts to write an approximate expression for the peak cross-section in the presence of Doppler broadening $\Delta\nu_{Doppler} >> \Delta\nu_{total\ radiative}$.