Estimation of the Lamb Shift in Hydrogen With First Order Perturbation Theory

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I presented an estimation of the Lamb Shift using first order perturbation theory.

We begain by considering a free electron under the influence of zero-point fluctuations in the electromagnetic field. First, we considered a mode ω of the electromagnetic field, and using the classical, non-relativistic, relation

$$m\frac{d^2\boldsymbol{x}}{dt^2} = e\boldsymbol{E}$$

obtained an expression for the mean square variation δx^2 in the electron's position from a single mode of the electromagnetic field.

As one does in such problems, we then summed over the modes of the electromagnetic field to obtain the total mean square variation δr^2 in the electron's position. Naively, one would sum over modes from 0 to ∞, but this integral is unfortunately infinite. To deal with this problem, we impose lower and upper cutoffs in the sum on the modes. We impose the upper cutoff at the frequency corresponding to the Compton wavelength of the electron because this is the scale at which we must consider relativstic effects. We impose a lower cutoff. A lower cutoff we impose at atomic frequencies because at this point our free electron model fails.

We then consider the electron in the Coulomb potential $V(r) = -\frac{e^2}{r}$. We expand the potential $V(\boldsymbol{r}+\delta\boldsymbol{r}) = [1 + \delta\boldsymbol{r}\cdot\nabla + \frac{1}{2}(\delta\boldsymbol{r}\cdot\nabla)^2 + ...]V(\boldsymbol{r})$.

Finally, we average $< V(\boldsymbol{r}+\delta\boldsymbol{r}) >$. δr has an isotropic spatial distribution, so the $\delta\boldsymbol{r}\cdot\nabla$ term vanishes in a spatial average. Thus, $< V(\boldsymbol{r}+\delta\boldsymbol{r}) >= [1 + \frac{1}{6} < (\delta r) >^2 \nabla^2 + ...]V(\boldsymbol{r})$. This average potential is the effective potential under which the electron is influenced, and we can treat this second term as a perturbation, by using the δr we obtained above by zero-point energy considerations. Using then first order perturbation theory, we then arrive at the following expression for the Lamb shift:

$$\Delta E = \frac{4e^2}{3}\alpha^3 a_0^2 \ln(\frac{mc^2}{Ry})|\psi(0)|^2$$

This energy shift vanishes for wavefunctions which have zero probability density at the nucleus–thus, this effect arises from the fact that the electron spends some of its time at the nucleus.

References

This talk was derived largely from, and followed closely:

Phys Rev. **74**, 1157 (1948): Welton "Some Observable Effects of the Quantum-Mechanical Fluctuations of the Electromagnetic Field."

Other portions of this talk were derived from lecture notes in Prof. Robert Littlejohn's Physics H190 class, Spring 2009.